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Enriched Polynomial Functors

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Abstract

The primary goal of this project was to define a new bicategory $\mathscr{V}$-**Poly** whose 1-cells are an enriched analogue of a recently studied categorical notion of polynomials, and guided by the idea that monads in $\mathscr{V}$-**Poly** should be Σ-free symmetric multicategories (also known as coloured operads). To reach this goal, we first discuss generalisations of polynomials within a category theoretic framework. We then discuss various well-established bicategories before embarking upon a definition of $\mathscr{V}$-**Poly**.

1 An introduction

We are all familiar with expressions of the form $2X^3 + 4X^2 + 9$, where the variable or indeterminate X is some element from a field $\mathbb{F}$ or ring. More generally, a polynomial has the form

$$a_nX^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \ldots + a_2X^2 + a_1X + a_0$$

where each a_i is a constant. Alternatively, the above may be re-written as $\sum_{i=0}^{n} a_iX^i$ using the sigma notation.

However, we may also think of polynomials as functions mapping X from some domain to its value in the codomain. For example,

$$P : \mathbb{R} \to \mathbb{R}, \quad P(X) = 2X^3 + 4X^2 + 9$$

is a polynomial function.

We may further extend this to cases where X is some 2×2 matrix (with entries over $\mathbb{F}$), and the coefficients themselves are also 2×2 matrices. For example,

$$P : \mathbf{M}_2 \to \mathbf{M}_2, \quad P(X) = \begin{pmatrix} 2 & 4 \\ 3 & \mathrm{e} \end{pmatrix} X^2 + \begin{pmatrix} 0 & 7 \\ \frac{1}{9} & \sqrt{2} \end{pmatrix}$$

sends a 2×2 matrix to another 2×2 matrix. But can we generalise this further still? That is, would it be possible to replace the indeterminate X by more abstract mathematical structures like groups or topological spaces? The answer is yes, but let us first consider the case where both indeterminates and coefficients are sets.

2 Polynomial functions and sets

For fixed sets A, B, C, N and R, along with our indeterminate X, a typical example[1] of a polynomial function $P(X) : \mathbf{Set} \to \mathbf{Set}$ is the following

$$P(X) = A + B \times X^N + C \times X^R$$

where X^N denotes the set[2] of all functions from N to X, "×" is the usual cartesian product of sets and "+" represents the disjoint union of sets. Now for some fixed mapping $p : E \to B$, we may also write a polynomial function in more general form[3] as follows

$$P(X) = \sum_{b \in B} \prod_{e \in E_b} X \tag{2.1}$$

where E_b is the fibre of p over b. That is, $E_b = \{e \in E : p(e) = b\}$ (a set indexed by an element $b \in B$).

Example 1. *Let $B = \{1, 2, 3, 4\}$, and suppose the fibres of p over each element in B are $E_1 = \{a_1, a_2\}, E_2 = \{b_1, b_2, b_3\}, E_3 = \varnothing$ and $E_4 = \{d_1, d_2\}$. Now since E_1 has two elements, we have $\prod_{e \in E_1} X = X \times X$. However, we may also think of $X \times X$ as the set of all functions*

[1] Example taken from Kock's set of notes on Polynomial Functors, p1

[2] The reason for this notation is because $\left|X^N\right| = |X|^{|N|}$. For example, if $N = \{1, 2\}$ and $X = \{a, b, c\}$, then a function $f : N \to X$ is a set $\{(1, b), (2, c)\}$. Since there are 3×3 different such functions, we have $\left|X^N\right| = 9 = |X|^{|N|}$.

[3] Formula adapted from the more general case of polynomials with multiple input and output variables. See Gambino and Kock [2010] p7

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from a set with only two elements to X, namely the set X^2. That is, $X \times X \cong X^2$. Therefore,

$$\begin{aligned} P(X) &= X^2 + X^3 + X^{\varnothing} + X^2 \\ &= X^2 + X^3 + 1 + X^2 \\ &\cong X^3 + 2 \times X^2 + 1 \end{aligned}$$

where we denote $X^{\varnothing}$ by 1 as there is only one function from the empty set to X. Note also that $X^2 + X^2 \cong 2 \times X^2$ since the disjoint union $X^2 + X^2$ gives us precisely two copies of X^2.

So the above polynomial P does indeed map a set X to another set $X^3 + 2 \times X^2 + 1$. In the language of category theory, we say that P is a functor from **Set** to **Set** (the category of all small sets).

3 A more abstract view on polynomials

While equation (2.1) defines a mapping from sets to sets, it is beneficial to present a different view of the same process to allow for cases where X may not be a set. The basic idea behind this view is outlined below.

Example 2. *Consider the functor P : **Set** $\rightarrow$ **Set** defined by $P(X) = X^3 + 2 \times X^2 + 1$ in the previous example. We may think of the final result as a process involving the following steps.*

[1] *Begin with a set $I = \{X\}$ containing only the set X (a set of sets).*

[2] *Let $E = \{X_1, X_2, \dots, X_7\}$ be a set and consider a mapping s from E to I. The fibre of s over X is the entire set E, and we can think of this as creating 7 copies of X.*

[3] *Next, let $B = \{X^3, X^2, X^2, 1\}$ and consider the mapping $p : E \rightarrow B$ which sends X_1, X_2, X_3 to X^3, X_4, X_5 to X^2 and X_6, X_7 to X^2. Denote by 1 those elements in B whose pre-image under p is empty.*

[4] *Finally, let t be a mapping from B to $J = \{X^3 + 2 \times X^2 + 1\}$ (where again J is a one element set).*

In other words, the above process may be represented by the data

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

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We define diagrams of the above type to be a polynomial, and the associated polynomial functor as the composite $\Sigma_t \Pi_p \Delta_s$ from the following diagram

$$\mathbf{Set}/_I \xrightarrow{\Delta_s} \mathbf{Set}/_E \xrightarrow{\Pi_p} \mathbf{Set}/_B \xrightarrow{\Sigma_t} \mathbf{Set}/_J$$

where Δ_s denotes a pullback along s, Π_p is a distributivity pullback[4] around p, and Σ_t denotes post-composition with t.

So why has a polynomial functor been defined this way? Consider an object in the slice category $\mathbf{Set}/_I$, a map $f : X \to I$ where $I = \{*\}$ is a singleton. Note that since I contains only the one element, there is a canonical isomorphism between f and X (regarded as the fibre of f over $*$). Pulling back along s gives us an object $E \times X \to E$ in $\mathbf{Set}/_E$. Since pullbacks are not unique, take $E \times X \to E$ to be the trivial mapping $(e, x) \mapsto e$. Like before, we may define a map $X \times E \to E$ by the collection of its fibres over each element $e \in E$. This gives us

$$(E \times X)_{e \in E} \cong \{e\} \times X \cong X$$

The functor $\Pi_p : \mathbf{Set}/_E \to \mathbf{Set}/_B$ is defined by $\Pi_p(Z) = \sum_{b \in B} \prod_{e \in E_b} Z_e$[5]. To see that $\sum_{b \in B} \prod_{e \in E_b} Z_e$ belongs to $\mathbf{Set}/_B$, first consider the product $\prod_{e \in E_b} Z_e$. This is the fibre over $b \in B$ of some map $\sum_{b \in B} \prod_{e \in E_b} Z_e \to B$. Hence, taking the disjoint union over all $b \in B$ gives us a family of cartesian products indexed by B and so $\Pi_p(Z) \in \mathbf{Set}/_B$.

Therefore applying the functor to $E \times X$, we get

$$\Pi_p(E \times X) = \sum_{b \in B} \prod_{e \in E_b} (E \times X)_{e \in E} \cong \sum_{b \in B} \prod_{e \in E_b} X$$

Finally, if J is also a singleton, the functor Σ_t simply maps the family of sets $\sum_{b \in B} \prod_{e \in E_b} X$ indexed by B to itself (regarded as a set with only one element $\sum_{b \in B} \prod_{e \in E_b} X$ or as the fibre of t over $*$ in J).

So in the case where both I and J are singletons, the polynomial functor $\Sigma_t \Pi_p \Delta_s$ is identical to the polynomial described by (2.1) (up to isomorphism). However, the idea

[4]See Weber [2011] for a definition of distributivity pullbacks.

[5]Definition taken from p30 of Kock's notes on Polynomial functors

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works even in the case where I and J are not singletons but are finite sets. The result is a polynomial function with more than one input variable and one output variable.

Although the discussion so far has restricted our objects I, E, B and J to be elements of the category **Set**, the way we have defined a polynomial and its associated functor allows for more general categories $\mathscr{E}$ (where $\mathscr{E}$ has all pullbacks and Δ_p has a right adjoint). And so a polynomial consists of objects $I, E, B, J \in \mathscr{E}$ and arrows s, p, t as shown below

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

and the associated polynomial functor is the same composite $\Sigma_t \Pi_p \Delta_s$ (but with $\mathscr{E}/_I$ instead of $\mathbf{Set}/_I$)

$$\mathscr{E}/_I \xrightarrow{\Delta_s} \mathscr{E}/_E \xrightarrow{\Pi_p} \mathscr{E}/_B \xrightarrow{\Sigma_t} \mathscr{E}/_J$$

Note: Since the polynomial functor is completely specified by its polynomial, we shall refer to the polynomial itself as the polynomial functor as this fits in better with the idea of enrichment[6].

4 Enriched polynomial functors

Having now defined our polynomial functor as the diagram $I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$ where $I, E, B, J \in \mathscr{E}$, it turns out that we may define a new bicategory $\mathbf{Poly}_{\mathscr{E}}$ with the following data:

[1] the objects (or 0-cells) are the objects of $\mathscr{E}$,

[2] the arrows (or 1-cells) are diagrams of the form $I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$,

[3] and the arrows between arrows (or 2-cells) are a pair of arrows $\langle \phi_0, \phi_1 \rangle$ in $\mathscr{E}$ such that the following diagram commutes

[6] See entry on Polynomial Functors at `http://ncatlab.org/nlab/show/polynomial+functor` where the polynomial itself is used as a synonym for the functor.

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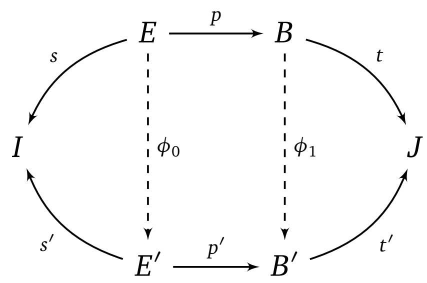

and the middle square is a pullback square.

The composition of such 1-cells has been extensively studied by Gambino and Kock[7], and Weber[8] but due to its complexity, the reader is referred to the relevant papers for a definition.

However, the key point to note is that if p is the identity arrow, then $\mathbf{Poly}_{\mathscr{E}}$ reduces to the well known bicategory $\mathbf{Span}_{\mathscr{E}}$. It is also quite well established that a monad in the bicategory $\mathbf{Span}_{\mathscr{E}}$ is an internal category in $\mathscr{E}$.

Another well known bicategory is that of $\mathscr{V}$-**Mat**. Consider a span from I to J as denoted by the diagram $I \overset{s}{\longleftarrow} E \overset{t}{\longrightarrow} J$ (where $I, E, J \in \mathbf{Set}$). Let

$$\mathrm{Hom}_{\mathbf{Set}}(i,j) = M(i,j) = \{e \in E : s(e) = i, t(e) = j\}$$

Now suppose $M(i,j)$ is no longer a set but an *object* in some monoidal category $\mathscr{V}$, and define the composition of two such objects $M(i,j)$ and $M(j,k)$ as follows

$$M(i,k) = \coprod_{j \in J} M(i,j) \otimes M(j,k)$$

where $\otimes$ is called a *tensor product*. The end result is the bicategory $\mathscr{V}$-**Mat**, and as it turns out, a monad in this bicategory is a $\mathscr{V}$-Category (or a category enriched in the monoidal category $\mathscr{V}$).

We are now in a position to be able to state the original aim of this project; to define a bicategory $\mathscr{V}$-**Poly** such that a monad in this bicategory should form a Σ-free symmetric multicategory[9].

[7] See Gambino and Kock [2010] p9

[8] See Weber [2011] p16

[9] See `http://ncatlab.org/nlab/show/symmetric+multicategory` for a definition of a symmetric multicategory. Here, Σ-free means that if $\mathscr{L}$ is the set of all linear orders on the objects $i_1, \ldots, i_n$,

5 The beginnings of a definition

Suppose a polynomial $I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$ is given, where each of I, E, B and J are sets and p has finite fibres (i.e., the fibre of p over b is finite for all $b \in B$). Consider a $b \in B$ such that $t(b) = j$ for some $j \in J$, and take its fibre over b to form the set $p^{-1}(b) = \{e_1, \ldots, e_n\}$. Now for each e_i in the set $p^{-1}(b)$, applying the mapping s results in another set $\{s(e_1), \ldots s(e_n)\} = \{i_1, \ldots, i_n\}$. Finally, placing some linear order on the set $\{i_1, \ldots, i_n\}$ gives an ordered sequence $(i_1, \ldots, i_n)$.

In other words, for an arbitrary $b \in B$ and some linear order on $\{i_1, \ldots, i_n\}$, we may define a new hom-set as follows

$$\mathrm{Hom}_{\mathbf{Set}}((i_1, \ldots, i_n); j) = M((i_1, \ldots, i_n); j) = \left\{b \in B : t(b) = j, s\left(p^{-1}(b)\right) = (i_1, \ldots, i_n)\right\}$$

with the additional requirement that there exists a bijection between $M((i_1, \ldots, i_n); j)$ and $M\left((i_{\varrho_1}, \ldots, i_{\varrho_n}); j\right)$ for any ϱ (a permutation on the linear order).

Now consider $M((i_1, \ldots, i_n); j)$ to be an object in a monoidal category $\mathscr{V}$ equipped with some tensor product $\otimes$. If we are able to describe the composition of two such objects, then we will most certainly have described our bicategory $\mathscr{V}$-**Poly** (or come close to doing so). Unfortunately, due to time constraints, this last bit of detail (and perhaps the hardest step in the project) will be left as a future endeavour.

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then $\varrho(\ell) = \ell$ for some $\ell \in \mathscr{L}$ implies that ϱ is the identity in S_n (the symmetric group on a set of n objects).

6 References

[1] S. Mac Lane. *Categories for the Working Mathematician*, *Graduate Texts in Mathematics*, volume 5. Springer, 1997

[2] G.M. Kelly. *Basic Concepts of Enriched Category Theory*, *Lecture Notes in Mathematics* volume 64. Cambridge University Press, 1982. Republished in *Reprints in Theory and Applications of Categories*, no. 10. 2005

[3] J. Kock. *Notes on Polynomial Functors*, *http://mat.uab.es/ kock/cat/notes-on-polynomial-functors.html*, 2009

[4] N. Gambino and J. Kock. Polynomial functors and polynomial monads. arXiv:0906.4931, 2010

[5] M. Weber. Polynomials in Categories with pullbacks. arXiv:1106.1983v1, 2011

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