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Theory of Poisson Point Process and its Application to Traffic Modelling

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Abstract

Poisson point process (or Poisson process) has important implications and wide applications in one and more dimensions. Kingman (1993) provides the main theorems and properties of Poisson process. Referring to his book I will give a brief review of those fundamental properties in the general settings on a Polish space in section 2. The idea of Poisson Laplace Functional given in Serfozo (2009) is introduced.I will then relate it to another important tool called Campbell's Theorem in section 2.2. Using these fundamental properties, I extend the simple road traffic model given by Kingman (1993) to a model regarding a number of vehicles involved in road tolls as a Poisson process in section 3. Following this, I will provide an extension using the technique of product space representation of the process. In the last section we employ the technique of Pearson's chi-squared goodness-of-fit test to perform statistical inference. I conclude that the reliability of modelling the number of vehicles in road tolls is a Poisson process.

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Preface

AMSI Vacation Research Scholar Experience

I I can now appreciate how the development of modern probability theory and stochastic processes sits comfortably between Pure Mathematics (Measure Theory in particular) and Statistical Sciences. During the course of the project I learnt that one needs to do a substantial level of reading, writing and thinking in order to get to a reasonably serious level of understanding and research in Mathematics and Statistics. This experience has provided a glimpse at the academic research side of Mathematics and Statistics. It is an invaluable opportunity invaluable and and has significantly strengthened my eagerness to pursue further study. The CSIRO Big Day In at Macquarie University is definitely an irreplacable experience as part of my undergraduate life. It is so exciting and rewarding to meet like-minds from all around Australia and listen to their presentations. As a final comment I strongly recommend this vacation research scholarship program to fellow students interested in Mathematics and Statistics.

Acknowledgements

I will close the preface section with a few acknowledgements. First and foremost, I would like to thank my supervisor A/Prof. Aihua Xia for his patient guidance during the course of the research project. Also, for his insightful advice on this research paper even after the formal period of vacation research scholarship had ended. Without him, this project would not be possible. I am also grateful to all my mentors in Mathematics and Statistics at the University of Melbourne. I would also like to extend my gratitude to AMSI, the CSIRO and the University of Melbourne for the generous provision of funding and opportunities of undergraduate research. Last but definitely not least I would like to give special thanks to my family and friends for their constant support.

1 Introduction

The name of Poisson process originates from Poisson distribution, which is considered as the limiting case of a binomial distribution[1]. Formally, X is a Poisson random variable with parameter measure Λ if it takes only non-negative integer values n and has probability mass function

$$P(X = n) = \pi_n(\Lambda) := \frac{\Lambda^n e^{\Lambda}}{n!}, \quad n \in \mathbb{Z}_+ := \{0, 1, 2, ...\}. \tag{1}$$

For the sake of brevity, lengthy proofs are replaced by sketch of proofs to avoid overloading of technicality. Readers interested in pursuing proofs and derivations in detail will be directed to respective references. Kingman (1993) provides a concrete introduction of the theory of Poisson process in Kingman (1993). Section 2 of this paper gives a brief review of the main theorems of Poisson process stated in his book together with sketched proofs. Readers looking for proofs in detail are advised to consult Kingman (1993). These properties form an important basis for developing our model in section 3.

Now we state the first theorem in this paper, a powerful result which gives information about infinite sums and conditions about convergence of Poisson random variables.

Theorem 1.1 (Countable Additivity Theorem). *Consider independent Poisson random variables X_j each equipped with parameter μ_j, $j \in \mathbb{N}^+$. If $\sigma = \sum_{j=1}^{\infty} \mu_j$ converges, then $S = \sum_{j=1}^{\infty} X_j$ is a Poisson random variable with parameter σ and S converges almost surely[2]. However if $\sigma = \sum_{j=1}^{\infty} \mu_j$ diverges, then $S = \sum_{j=1}^{\infty} X_j$ diverges almost surely.*

Proof. Let $S_n = \sum_{j=1}^{n} X_j$. By mathematical induction it follows that S_n is a Poisson random variable with parameter $\sigma_n = \sum_{j=1}^{n} \mu_j$. Thus for any $r \geq 0$, $P(S_n \leq r) = \sum_{k=0}^{r} \pi_k(\sigma_n)$. Since for fixed r the events $\{S_n \leq r\}$ decreases with n, it easily follows that

$$P(S \leq r) = \lim_{n\to\infty} P(S_n \leq r) = \lim_{n\to\infty} \sum_{k=0}^{r} \pi_k(\sigma_n). \tag{2}$$

If σ_n converges to some finite σ, by continuity of the Poisson probability mass function, (2) becomes $\sum_{k=0}^{r} \pi_k(\sigma)$ so that $P(S = r) = \pi_r(\sigma)$ and hence S is a Poisson random variable with parameter σ. However if σ_n diverges, $\sum_{k=0}^{r} \pi_k(\sigma_n) =$

[1]The reader who is interested in pursuing the derivation of a Poisson probability mass function from a binomial one is recommended to read page 2 and 3 of Kingman (1993).

[2]An event $\mathcal{K}$ is said to occur almost surely when $P(\mathcal{K}) = 1$.

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$\exp(-\sigma_n)\sum_{k=0}^{r}\sigma_n^k/k! \to 0$ and implies $P(S > r) = 1$ for all r, hence S diverges almost surely and the proof is complete. □

2 Poisson Process

2.1 Definition and fundamental properties

In terms of the mathematical setting, we define the probability space $(\Omega,\mathcal{F},P)$ with $P(\Omega) = 1$, where Ω is often referred to as the sample space. Suppose there is a basic collection of measurable[3] sets of points $\omega \in \Omega$. The σ-algebra $\mathcal{F}$ is completed by sets of probability zero. a Polish space[4] S with the σ-algebra $\mathbf{A} = \mathcal{A}(S)$ of Borel subsets. The Polish space S is referred to as the state space. The Polish space S is usually a d-dimensional Euclidean space for some d, or more generally a manifold, which is *locally equivalent* to $\mathbb{R}^d$.

Let Π denote our Poisson process. A clear definition of what a Poisson process is will be given in next subsection. For any *test set* $A \in \mathbf{A}$ write the count function $N(A) : \Omega \to \{0, 1, ..., \infty\}$ as

$$N(A) = \#\{\Pi \cap A\}. \tag{3}$$

We require this to be a measurable mapping. A more sophisticated way of defining the count function is in form of a Dirac measure[5], that is,

$$N(A) = \sum_{x\in\Pi} \delta_x(A). \tag{4}$$

Now we have the background knowledge to state the definition of a Poisson process. While the count function N itself is often referred to as a Poisson process by many authors, for instance, Grimmett and Stirzaker (1991), Borovkov (2003) and Serfozo (2009), for the sake of generality we follow the construction suggested in Kingman (1993).

Definition 2.1. *A Poisson Process is a random countable subset $\Pi \subset S$, defined on a probability space $(\Omega,\mathcal{F},P)$, such that*

[3]Sets from the σ-algebra $\mathcal{F}$ are said to be measurable subsets of Ω. For a more precise mathematical setting of measurability of sets in detail, see page 19 in [4].

[4]A Polish space is a compelete separable metric space or more generally a completely separable metrisable topological space.

[5]A Dirac measure is a measure δ_x such that $\delta_x(A) = \mathbf{1}(x \in A)$, where $\mathbf{1}(.)$ is the *indicator function.*

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1. *for a finite choice of disjoint subsets $A_1, ..., A_n \subset \mathbf{A}$, $N(A_1), ..., N(A_n)$ are statistically independent,*

2. *and for all $A \subset \mathbf{A}$, $N(A)$ is a Poisson random variable with mean measure*[6] *$\mu(A) = \mathbb{E}[N(A)] \in [0, \infty]$. μ is called the mean measure of N.*

We assume that for any position there is no overlapping of points in Π. Hence it is important to remark that μ must satisfy a specific condition to be defined as a mean measure, as stated below.

Remark 2.2. *A mean measure must be non-atomic such that*

$$\mu(\{x\}) = 0, \quad \forall x \in S. \tag{5}$$

Proof. Assume not. Suppose the measure μ has an atom at $x \in S$, such that $\mu(\{x\})>0$. Then $P(N(\{x\}) \geq 2) = 1 - e^{-\mu(\{x\})} - \mu(\{x\})e^{-\mu(\{x\})} > 0$, a contradiction. □

Unless stated otherwise, throughout the discussion we assume μ is non-atomic in the sense of (5). When we refer the Polish space to some Euclidean space of dimension d, the mean measure μ is absolutely continuous with respect to the Lebesgue measure dx and given by some *rate* or *intensity function* $\lambda(.)$ as follow:

$$\mu(A) = \int_A \lambda(x)\; dx. \tag{6}$$

Here we say $\lambda(x)$ is location-dependent with respect to the measurable mapping $N(.)$. When $\lambda(x) = \lambda$, a constant, we say that the Poisson process is homogeneous. Note that dx in the sense of (6) is not necessarily one-dimensional.

2.2 Campbell's Theorem and Poisson Laplace functionals

In this section we introduce Campbell's Theorem which establishes a general master equation (9). Using Campbell's Theorem we obtain the Poisson Laplace functional, a very important tool for deriving properties and theorems of a Poisson process. Kingman (1993) gave the entire version of Campbell's theorem, leading to straightforward computation of expectations and variances of sum of Poisson process. We state the full version of Campbell's theorem, but only give a partial proof, which serves our purpose to establish the Poisson Laplace functional.

[6] It is sometimes called the parameter measure, parameter function, leading measure, first moment measure or mean measure in the statistical literature.

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Theorem 2.3 (Campbell's Theorem)**.** *Suppose* Π *is a Poisson process on a Polish space* S *with mean measure* μ *and suppose* $f : S \to \mathbb{R}$ *is a measurable mapping. Define the sum as*

$$\Sigma_f = \sum_{X \in \Pi} f(X). \tag{7}$$

It is absolutely convergent with probability if and only if

$$\int_S \min(|f(x)|, 1)\mu(dx) \tag{8}$$

is finite (or μ*-integrable). If such condition holds, then*

$$\mathbb{E}(e^{\theta \Sigma_f}) = \exp\left\{ \int_S (e^{\theta f(x)} - 1)\mu(dx) \right\} \tag{9}$$

for any $\theta \in \mathbb{C}$ *such that the integral converges. Also, the expectation*

$$\mathbb{E}(\Sigma_f) = \int_S f(x)\mu(dx) \tag{10}$$

is defined whenever the integral converges. If (10) *is finite then the variance*

$$Var(\Sigma_f) = \int_S f(x)^2 \mu(dx) \tag{11}$$

exists in the sense that it may be either finite or infinite.

For our purpose it suffices to give a partial proof of the theorem, showing that (9) holds in the case when $f \geq 0$. Readers interested in pursuing the entire proof in detail is advised to consult page 28 to 30 in Kingman (1993).

Partial proof. For $\theta \in \mathbb{C}$ and f simple[7], let $A_j = \{x; f(x) = f_j\}$ be disjoint and measurable with locally finite[8] mean measure $m_j = \mu(A_j)$. Then $N_j = N(A_j)$ are independent Poisson random variables with parameter m_j and $\Sigma_f = \sum_{X \in \Pi} f(X) =$

[7] f is called simple if it is a function that takes a finite number of values $f_1, ...f_n$ with finite μ-measures and vanishes outside a set of finite μ-measures

[8] A measure is called locally finite if every point of the measure space has a neigbourhood of finite measure.

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$\sum_{j=1}^{n} f_j N_j$. Now

$$\begin{aligned}\mathbb{E}\Big\{e^{\theta\Sigma_f}\Big\} &= \prod_{j=1}^{n}\mathbb{E}\Big\{e^{\theta f_j N_j}\Big\} \\ &= \exp\Big\{\sum_{j=1}^{n}\int_{A_j}(e^{\theta f(x)}-1)\mu(dx)\Big\} \\ &= \exp\Big\{\int_S(e^{\theta f(x)}-1)\mu(dx)\Big\}.\end{aligned}$$

Now we know that (9) holds for simple functions f. We are going to extend this to any positive measurable functions by integration theory. Consider the case when f is positive. Taking $\theta = -u$ real and negative, writing $\Sigma_{f_j} = \sum_{X\in\Pi} f_j(X)$ for $\{f_j\}$ being an increasing sequence of simple functions having limit f, we deduce that $\mathbb{E}(e^{-u\Sigma_f}) = \lim_j \mathbb{E}(e^{-u\Sigma_{f_j}}) = \exp\big\{\int_S(e^{-uf(x)}-1)\mu(dx)\big\}$ by Lebesgue's monotone convergence theorem. We know that if (8) holds, the integral on the right hand side converges showing that Σ_f is a finite random variable. However if (8) does not hold, the integral diverges and $\mathbb{E}(e^{-u\Sigma_f}) = 0$, indicating that Σ is infinite almost surely. So the theorem is proved for $f \geq 0$. □

Restricting Campbell's theorem to the condition that $f : S \to \mathbb{R}_+$ and setting $\theta = -1$, we now obtain the Poisson Laplace Functional[9], namely

$$\mathbb{E}\Big\{e^{-\Sigma_f}\Big\} = \exp\Big\{-\int_S(1-e^{-f(x)})\mu(dx)\Big\}. \tag{12}$$

We remark that the Poisson Laplace Functional uniquely defines a Poisson process.

2.3 Sum of Independent Poisson Processes

The Poisson process has a number of special properties which often makes calculations and derivations surprisingly neat and simple to analyse. We first look at the superposition theorem regarding the sum of independent Poisson processes, the proof of which requires a knowledge of the Countable Additivity Theorem and a lemma called the Disjointness Lemma. For sake of brevity we state the lemma without proof here. Readers interested in pursuing a detailed proof is advised to read page 15 in Kingman (1993). Below we assume A is a measurable set on S.

[9] A derivation of the Poisson Laplace Functional without using Campbell's Theorem is written on page 185 of Serfozo(2009).

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Lemma 2.4 (Disjointness Lemma). *If Π_j and Π_k are independent Poisson processes each equipped with locally finite non-atomic mean measure $\mu_j(.)$ and $\mu_k(.)$, then Π_j and Π_k are disjoint almost surely. Algebraically,*

$$P(\Pi_j \cap \Pi_k \cap A = \emptyset) = 1. \tag{13}$$

An obvious implication of the lemma is that n Poisson processes are disjoint on some measurable set as long as they are independent.

Theorem 2.5 (Superposition Theorem). *Consider independent Poisson processes Π_j each with locally finite mean measure μ_j. It follows that their superposition $\Pi = \bigcup_{j=1}^{\infty} \Pi_j$ is a Poisson process on S with mean measure $\mu = \sum_{j=1}^{\infty} \mu_j$.*

Proof. Let $N_j(A) = \#\{\Pi_j \cap A\}$. The Disjointness Lemma shows the independent sets Π_j are disjoint on A. Hence if each $\mu_j(A)$ is locally finite, we have

$$N(A) = \sum_{j=1}^{\infty} N_j(A). \tag{14}$$

By the Countable Additivity Theorem, $N(A)$ is a Poisson random variable with measure $\mu = \mu(A) = \sum_{j=1}^{\infty} \mu_j$. In addition, if for some j, $\mu_j(A)$ is not locally finite, again by the Countable Additivity Theorem $N_j(A) = N(A) = \infty$ with probability 1 and (14) holds trivially. Moreover, $N(A_j)$ are disjoint for each j and $N_j(A_n)$ are all independent for $j \in \mathbb{N}$ and $n = 1, ..., k$. Hence for disjoint A_n it follows that $N(A_n)$,$n = 1, ..., k$, are independent and the proof is complete. □

Theorem 2.5 immediately leads to the following corollary, giving information about a finite sum of independent Poisson processes.

Corollary 2.6. *Let Π_j, $j = 1, ..., n$, be independent Poisson processes each with locally finite mean measure μ_j. It follows that their superposition $\Pi = \bigcup_{j=1}^{n} \Pi_j$ is a Poisson process with mean measure $\mu = \sum_{j=1}^{n} \mu_j$.*

Proof. Following the proof of Theorem 2.5, for $k = n+1, n+2...$, take $\Pi_k = \emptyset$ and the result follows. □

2.4 Transformation of Poisson Processes

Another important theorem of Poisson process Π is its preservation of properties under measurable mapping (Kingman, 1993). If Π on the Polish space S is mapped onto

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another Polish space T, the mapped random points form a Poisson process under some specified conditions. For instance, we need μ to be σ-finite[10]. In a formal sense the theorem is stated as follows.

Theorem 2.7 (The Mapping Theorem)**.** *Consider a Poisson process* Π *on a Polish space* S *equipped with a* σ*-finite and non-atomic mean measure* μ*,* $\Phi : S \to T$ *a measurable function such that the induced mean measure* μ^* *on* T *is again non-atomic given our assumption of no overlapping of points. Then* $\Phi(\Pi)$ *is a Poisson process on* T *with mean measure* μ^**.*

Proof. (Sketch) We make use of the Disjointness Lemma and Superposition Theorem in the proof. Given the σ-finiteness condition, assume there exists disjoint S_i such that $S = \bigcup_{i=1}^{\infty} S_i$ with $\mu(S_i) < \infty$. Write Π_i as the restriction of Π to S_i. Then Π_i are independent Poisson process with locally finite mean measure μ_i. So are $\Phi(\Pi_i)$ with induced mean measure μ_i^*. The Disjointness Lemma immediately tells that $\Phi(\Pi_i)$ are almost surely disjoint on S. Hence by the Superposition Theorem, $\Phi(\Pi) = \Phi(\bigcup_i \Pi_i) = \bigcup_i \Phi(\Pi_i)$ is again a Poisson process with induced mean measure $\mu^* = \sum_i \mu_i^*$. The reader interested in pursuing a detailed proof is advised to read page 18 and 19 in Kingman (1993). □

2.5 Existence of Poisson Processes

We now move on to another important property which serves as a reverse argument of what we have covered earlier. Here we seek to show the existence of a Poisson process with given mean measure μ on a Polish space S.

Theorem 2.8 (The Existence Theorem)**.** *If* $\mu = \sum_{i=1}^{\infty} \mu_i$ *with locally finite* $\mu_i(S)$*, then a Poisson process exists on* S *with mean measure* μ*.*

Proof. (Sketch) We follow the construction given by Kingman (1993). Without loss of generality, construct positive mean measures μ_n and independent random variables N_n and ξ_{nk} where $n, k \in \mathbb{N}$ in the sense that N_n is a Poisson random variable with mean measure $\mu_n(S)$ and ξ_{nk} having distribution $p_n(.) = \mu_n(.)/\mu_n(S)$. Write $\Pi_n = \{\xi_{n1}, ..., \xi_{nN_n}\}$ and

$$\Pi = \bigcup_{n=1}^{\infty} \Pi_n. \tag{15}$$

[10] The σ-finiteness condition guarantees that the Polish space S can be written as a countable union $\bigcup_{i=1}^{\infty} S_i$ with $\mu(S_i) < \infty$.

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If we write $N_n(A) = \#\{\Pi_n \cap A\}$, for disjoint $A_1, ...A_k$, $A_0 = (\bigcup_{i=1}^k A_k)^c$, $m = \sum_{i=0}^k m_i$, by the Law of Total Probability, it can be shown that

$$P(N_n(A_1) = m_1, ..., N_n(A_k) = m_k) = \prod_{i=0}^{k} \pi_{m_i}(\mu_n(A_i)). \quad (16)$$

Hence it implies that $N_n(A_i)$ are independent random variables with distribution $Pn(\mu_n(A_i))$. Thus Π_n are independent Poisson process with non-atomic mean measure μ_n. By the Superposition Theorem, (15) shows Π is a Poisson process with mean measure μ. The reader interested in pursuing a detailed proof, in particular the derivation of (16), is advised to read page 24 in Kingman (1993). □

2.6 The Colouring Theorem

We will discuss two simple, yet very useful, results of marked Poisson process which underpin the construction of our road toll traffic model. The first one is the Colouring Theorem, while the other is the Marking Theorem. As before, we denote our Poisson process Π having locally finite mean measure μ on a Polish space S. We construct as follows. Assume we have z different colours with z being a finite positive integer. If the points on a Poisson process independently receives the j^{th} colour with probability p_j, we get to the Colouring Theorem by writing Π_j as the set of points with the j^{th} colour.

Theorem 2.9 (The Colouring Theorem). *The Π_j are independent Poisson processes with mean measures $\mu_j = p_j\mu$.*

Proof. (Sketch) We prove by induction. First we start with $z = 2$. For any $A \subset S$, let $N(A)$ denote the number of points in A, $N_1(A)$ the number of points with 1^{st} colour (with probability p_1) in A, $N_2(A)$ the number of points with 2^{th} colour (with probability p_2) in A such that $\sum_{i\in\{1,2\}} p_i = 1$. Now $N(A)$ is a Poisson random variable with parameter $\mu(A)$. It follows that $N_1(A)$ and $N_2(A)$ (or $N(A) - N_1(A)$) are independent Poisson random variables with parameters $p_1\mu(A)$ and $p_2\mu(A)$ respectively [11]. By mathematical induction on the number of colours to any finite number z the theorem follows. □

[11] The reader interested in pursuing the proof of this statement can read page 163 and 164 of Borovkov(2003).

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2.7 Marked Poisson Processes

In this section we introduce the Marking Theorem. The Marking Theorem serves as a generalisation of the Colouring Theorem, as the latter is often too restrictive at the time of application. In order to prove the theorem, we first recall the Poisson Laplace Functional:

$$\mathbb{E}\left\{e^{-\Sigma_f}\right\} = \exp\left\{-\int_S (1 - e^{-f(x)})\mu(dx)\right\}. \tag{17}$$

Now suppose for each $x \in X$ on Π, we give a mark $m_x \in M$, where M is some measurable space called the marking space. The mark of x is independent to each other with different x. In terms of the pair (X, m_X), we obtain a random countable subset

$$\{(X, m_X); X \in \Pi\} = \Pi^* \subset S \times M. \tag{18}$$

Now let $p : S \times M \to [0, 1]$ be a probability kernal[12]. The fundamental result of the Marking Theorem is as follows.

Theorem 2.10 (The Marking Theorem)**.** *The random countable subset* Π^* *is a Poisson process on the product space* $S \times M$ *with mean measure given by*

$$\mu^*(C) = \iint_{(x,m_x)\in C} p(x, dm)\mu(dx). \tag{19}$$

Proof. (Sketch) Again we follow the construction given in Kingman (1993). Write

$$\Sigma^* = \sum_{X\in\Pi} f(X, m_X) \tag{20}$$

$$f_*(x) = -\log \int_M \exp\left\{-f(X, m_X)\right\} p(X, dm) \tag{21}$$

and replace Σ_f and $f(x)$ in our Poisson Laplace functional (17) by (20) and (21) respectively to compute[13]

$$\begin{aligned}\mathbb{E}\left\{e^{-\Sigma^*}\right\} &= \exp\left\{-\int_S\int_M \{1 - e^{-f(x,m_x)}\}\mu(dx)p(x, dm_x)\right\}\\ &= \exp\left\{-\int_{S\times M} (1 - e^{-f})d\mu^*\right\},\end{aligned}$$

[12] A probability kernel is a probability measure $p(x, .)$ on M for $x \in S$ such that $p(., B)$ is a measurable function on S for $B \subset M$.

[13] The reader who intends to pursue the detailed proof, in particular a detailed derivation of (21) and $\mathbb{E}(e^{-\Sigma^*})$ should read page 56 of Kingman (1993).

showing that Π^* follows a Poisson process with mean measure μ^* given by (19). □

The theorem immediately leads to the following corollary.

Corollary 2.11. *The marks of X, m_X, is a Poisson process on M.*

Proof. This is true because of the mapping theorem. The non-atomic mean measure of m_X can be computed by putting $C = S \times B$ in (19):

$$\mu_m(B) = \int_S \int_B p(x, dm)\mu(dx). \tag{22}$$

□

The theorem also suggests a generalisation of the Colouring Theorem by allowing the probabilites of colouring to change with respect to different values of x. As assumed in section 2.6, if there are $z < \infty$ different marks, the theorem implies that the points coloured with the i^{th} mark is a Poisson process (independent with other i's) with non-atomic mean measure

$$\mu_i(A) = \int_{x \in A} p(x, \{m_i\})\mu(dx). \tag{23}$$

Remark 2.12. *The probability of colouring now varies with x. For the i^{th} mark, $p_i = p(x, \{m_i\})$.*

3 Application to Traffic Models

In this section, we use the theorems and properties developed in previous paragraphs to establish the road toll model as a Poisson process. We extend the simple one-dimensional model of the number of cars as a Poisson process given by Kingman (1993).

3.1 Poisson Road Toll Model

Let S, the state space, be the real line $\mathbb{R}$ that represents the road and the points repesent each car. We construct the model as follows. Observe that the distribution of cars is not fixed with time because cars are moving with different speed. Taking a snapshot of the road $\mathbb{R}$ at some time point t, it may be realistic to model the cars

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as a Poisson process. An interesting question will be whether the cars form a Poisson process at some later times t, given that they form a Poisson process at $t = 0$. To solve the question, Kingman (1993) suggests we assume the speeds of the cars are random variables, independent of one another and their position on the road. We also construct a finite set of speeds a vehicle involved in road tolls can take

$$\mathcal{S} = \{s_1, s_2, ..., s_k\}, \qquad \text{where } s_1 < s_2 < ... < s_k \tag{24}$$

Let p_j be the probability that a vehicle involved in road tolls moves at a speed of s_j. Thus we obtain the finite set of respective probabilities:

$$\mathcal{P} = \{p_1, p_2, ..., p_k\}, \qquad \text{where} \sum_{j=1}^{k} p_j = 1 \tag{25}$$

We also assume overtaking is unrestricted. From a practical point of view, the statement above assumes overtaking is uninhibited. One may think of freeways and highways as a real world example. At this point the model is built, where statistical evidence will be presented in the following section. We solve the problem step by step using the properties we have derived in previous sections. It turns out that we can treat the solution of this question as an application of the Colouring Theorem, the Mapping Theorem, the Disjointness Lemma and the Superposition Theorem.

Write Π_j as the set of cars moving at speed s_j in $t = 0$ with a positive measurable intensity function $p_j\lambda(x)$. By the finiteness assumption of the set of possible speeds, the Colouring Theorem guarantees that Π_j are independent Poisson processes for each $j = 1, ..., k$. One reason why we partition the set in terms of respective speed, instead of our properties of the vehicles, is that we can explicitly establish a mapping of the Poisson process from period i of year Y to some time later as follows. At time t the position of the cars form a Poisson process $\Pi_j + s_j t$ obtained by translating Π_j by $s_j t$. We create a measurable mapping $\Phi : \mathbb{R} \to \mathbb{R}$ in the sense that

$$\Phi\big\{\Pi_j\big\} = \Pi_j + s_j t = \Pi_j(t). \tag{26}$$

Now in accordance with the Mapping Theorem, the number of vehicles with speed s_j involved in road tolls at t, $\Pi_j(t)$, again forms a Poisson process. Since for each $j = 1, ..., k$, $\Pi_j + s_j t$ is an independent Poisson process, again by the Disjointness Lemma we conclude that they are disjoint. Applying the Superposition Theorem, we see that the cars at a later time t, denoted by

$$\Pi(t) = \bigcup_{j=1}^{k} \Pi_j(t) = \bigcup_{j=1}^{k} \Big\{\Pi_j + s_j t\Big\}, \tag{27}$$

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is again a Poisson process. In terms of the rate of $\Pi(t)$, we see that each $\Pi_j + s_j t$ admits a rate of $p_j\lambda(x - s_j t)$. Hence the second part of the Superposition Theorem implies that $\Pi(t)$ has rate

$$\lambda_t(x) = \sum_{j=1}^{k} p_j \lambda(x - s_j t). \tag{28}$$

Now if we assume the probability of accidents is some $\beta > 0$ (which should be very small), applying the Colouring theorem again, we see that the traffic accidents at time t again follow a Poisson process.

An immediate and significant implication of (28) is that if the cars admit a homogenous Poisson process with some constant parameter λ at time 0, they form again a homogenous Poisson process at some later time t. As the state space is $\mathbb{R}$, the process being homogenous implies that it is also a renewal process whose inter-arrival times are independent exponentially distributed with parameter λ[14].

3.2 Extension by Product Space Representation

It is often desirable to obtain a simple model, so calculations can be neat. However the simple model presented above has several obvious limitations. Firstly, we should accept that the finiteness assumption of the set of possible values of speed does not hold true in practice. We can improve this by making the difference between each s_j very small and giving approximation of all possible speeds in practice. Furthermore, the assumption that the speed remains constant at all times and independent of the position of the vehicle is not realistic. Therefore it is natural to start with a product space representation similar to (18) used in the Marking Theorem which improves the generality of the model and eliminates the limitations stated above. This is suggested in Kingman (1993). In section 3.1, we first require the assumption that the speeds of vehicles involved in road tolls are independent random variables with respect to speeds of others and their respective position on roads. By means of the product space representation, we can now partially relax these assumptions. All we need to assume is the independence of movements of different vehicles. This should be a realistic assumption on freeways and highways where the road is relatively wide and has no restriction of overtaking (in most instances). Suppose the vehicles involved in road tolls form a Poisson process Π with intensity $\lambda(x)$ at time 0. In order to model the position of those vehicles at some later time t as a Poisson process $\Pi(t)$, we assume

[14]The proof of this fundamental property is written on page 173 and 174 in Serfozo(2009).

that the movements of vehicles involved in road tolls are independent. Write $\Psi_t(X)$ as the position at some time t. Under such construction the position of the car at some later time depends on current position. Now in accordance with the assumption that $\Psi_t(X)$ are independent for each X, as in section $2.7, \Psi_t(X)$ forms a marking of $\Pi(t)$ and also by the Marking Theorem the pair $(X, \Psi_t(X))$ forms a Poisson process $\Pi^*(t)$ with the product space representation

$$\Pi^*(t) = \{(X, \Psi_t(X)); X \in \Pi\} \subset \mathbb{R}^2. \tag{29}$$

Now suppose $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a measurable mapping such that

$$\Phi(X, \Psi_t(X)) = \Psi_t(X). \tag{30}$$

By the Mapping Theorem we deduce that $\Psi_t(X)$ forms a Poisson process $\Pi(t)$. Again if we assume the probability of accidents is some $\beta > 0$ (which should be very small), applying the Colouring theorem again, we see that the traffic accidents at time t again follow a Poisson process. Such construction efficiently removes the assumption that the speed s_j remains constant for all time. Now, in accordance with Remark 2.12, the probability that a specific speed is taken now varies with the vehicle's specific position. Finally, we remark that this model requires that the movement of vehicles is independent. If this assumption does not hold true the Marking Theorem would not follow. More serious is that it violates the basic definition of a Poisson random variable.

3.3 Superposition of the Model in the Case of Australia

One may notice the need of the Poisson road toll model to be equipped with different rate functions, possibly due to differences in situations such as development, technology and driving education. I will give an extension of the model in relation to Australian geography, however it should be noted that this also applies to other countries. . For the sake of convenience, we use short form to identify different states and territories in Australia. For instance, V denotes Victoria, NSW denotes New South Wales, Q denotes Queensland, WA denotes Western Australia, SA denotes South Australia, NT denotes Northern Territory and finally T denotes Tasmania. Let α be an element of the set $\mathbb{A} = \{V, NSW, Q, WA, SA, NT, T\}$. Assume at time 0 the number of vehicles involved in road tolls in α follows a Poisson process Π_α with rate $\lambda_\alpha(x)$. It may be reasonable to assume that for each $\alpha \in \mathbb{A}$, Π_α are independent Poisson processes. Now by the Disjointness Lemma we deduce that for each $\alpha \in \mathbb{A}$, Π_α is disjoint. Hence we can apply the Superposition Theorem and conclude that at some later time t the cars in all area of Australia $\Pi_{AU}(t)$ follow a Poisson Process

$$\Pi_{AU}(t) = \bigcup_{\alpha \in \mathbb{A}} \Pi_\alpha(t) \tag{31}$$

with rate

$$\lambda_{AU}(x) = \sum_{\alpha \in \mathbb{A}} \lambda_\alpha(x). \tag{32}$$

3.4 Possible Future Development and Research

In this section I will provide possible extensions or future developments for this model. Kloeden, McLean and Glonek (2002) give a logistic model between relative risk of casualty crash involvement (RR) and a given free travelling speed (v):

$$RR(v) = \exp\{-0.822957835 - 0.083680149v + 0.001623269v^2\} \tag{33}$$

By standard computation the first order derivative of $RR(v)$ equals

$$RR'(v) = (-0.83680149 + 0.3246538v)\exp\{-0.822957835 - 0.83680149v + 0.1623269v^2\} \tag{34}$$

It follows that $RR(v)$ is an increasing function of v for $v > 25.77519468$. This relation aligns with the example given by Kloeden et. al. that the relative risk is 3.6 times greater for a vehicle travelling in a 60 km/hr zone at a speed of 70 km/hr than at a speed of 60 km/hr[15]. In addition, research by TAC (2002) shows that in a metropolitan area the risk of of being involved in a crash increases exponentially. Based on these research results, one may want to exploit the relationship between the number of vehicles involved in road tolls in a given time period $N(A_t)$ for some time t and our set of possible speeds in (24), in particular the upper limit s_k. Intuitively speaking, injuries and road tolls may also occur when a proportion of road users drive at a slower speed alongside others who are driving at a higher speed. This suggests that one may also want to exploit the relationship between the lower limit s_1 of the set $\mathcal{S}$ and $N(A_t)$. A rigorous model and mathematical relation between elements of the set $\mathcal{S}$ and $N(A_t)$ therefore needs to be established.

The assumption that we need movements of vehicles to be independent from others

[15] Kloeden et. al. give detailed reasons in their paper and state that the curve should not be interpreted literally below $v = 25.77519468$ where the slope changes sign.

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may be realistic in rural areas, but it does not apply to the Central Business District. Within the Central Business District it should be accepted that overtaking is inhibited and movements of vehicles are not independent. Vehicles move in a way dependent to each other. Dependence between movement of vehicles is often a crucial factor needed to be taken into account. We have made several important assumptions for the sake of simplication and consistency of Poisson. In a strict sense, the Poisson process may not be a sufficient model as the independence assumption is often not satisfied. Thus traffic modelling has become an active research topic in recent decades, using various mathematical techniques such as graph theory, stochastic modelling and statistics.. A challenge that still remains is to build a more concrete mathematical model to better simulate and predict the real world settings and environment.

4 Statistical Inference of Poisson Process

In this section we test the reliability of a Poisson Model on road tolls. Recall the definition of a Poisson process as stated in definition 2.1. We construct a reasonable and suitable set of disjoint subsets by restricting the time interval from year 2008 to 2012. Observations are confined within Victoria, where the age of road users stays between 18 to 25. This is to ensure a similar set of driving skills and experience. Suppose $A_1 = \text{January}, A_2 = \text{February}, ..., A_{12} = \text{December}$, $N(A_i)$ is the number of vehicles involved in road tolls in month A_i. It is clear that A_i are disjoint. It is almost impossible to check the independence condition within the first section of the Poisson process definition. Having said that, we move on to the statistical inference of the second half. We want to examine whether $N(A_i)$ follows a Poisson random variable in the traffic model. In particular, using the road statistics provided by Vicroads, we perform the Pearson's chi-squared goodness-of-fit test. In short, the Pearson's chi-squared goodness-of-fit test involves the comparison between the chi-squared statistic χ^2 and a critical value $\chi^2_\alpha(k-1)$, where $k-1$ denotes the degree of freedom and α is the desired significance level of the test. The chi-squared statistic is computed by

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \sum_{i=1}^{k} \frac{(O_i - np_i)^2}{np_i}, \tag{35}$$

where O_i denotes the observed frequency, E_i expected frequency, n the total number of events, p_i the probability of the i^{th} event. Throughout, we test the null hypothesis that our distribution follows Poisson. Hence under level of significance α, we reject the null if and only if

$$\chi^2 > \chi^2_\alpha(df), \quad \text{where } df \text{ is the degree of freedom.} \tag{36}$$

Data regarding the number of monthly road tolls of age 18-25 in the Melbourne area from 2008 to 2009 is depicted in the table below:

Month	Oberved Frequency O_i
January 2008	5
February 2008	2
March 2008	7
April 2008	8
May 2008	10
June 2008	6
July 2008	6
August 2008	6
September 2008	4
October 2008	6
November 2008	6
December 2008	10
January 2009	12
February 2009	2
March 2009	5
April 2009	5
May 2009	5
June 2009	11
July 2009	3
August 2009	4
September 2009	2
October 2009	6
November 2009	4
December 2009	4

For a set of data $\{X_1, ..., X_n\}$, the maximum likelihood estimator for the mean measure of a Poisson distribution is its $\bar{X}/n$, where $\bar{X} = \sum_{i=1}^n X_i$. Hence based on the data presented in the above table, the maximum likelihood estimator $\hat{\mu}$ of μ is $141/24$ provided it follows a Poisson distribution. Now we test the null hypothesis $N(A_i)$ is a Poisson random variable with mean $141/24$ versus the alternative hypothesis that $N(A_i)$ is not a Poisson random variable.

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Remark 4.1. *It is important to keep in mind we do not prove it is a Poisson random variable. What we show with hypothesis testing is that oue data does not violate the possbility that it follows Poisson.*

Partition the observations into sets $B_1 = \{0, 1, 2, 3, 4\}$, $B_2 = \{5, 6\}$, $B_3 = \{7, 8, 9, 10, ...\}$[16]. We compute the respective probabilities of each set occuring and the expected number of road tolls E_i as follow:

-	B_1	B_2	B_3
Observed Frequency O_i	8	10	6
Probability $P(B_i)$	0.3021365	0.3242354	0.3736281
Expected Frequency E_i	7.251276	7.78165	8.967074

Using (35) and information from the table above, it follows that the chi-squared test statistic $\chi^2 = 1.691465$. Since we estimated $\hat{\mu} = 141/24$, χ^2 has an approximate chi-squared distribution of $3 - 1 - 1 = 2$ degrees of freedom. Now under 5% level of significance,

$$\chi^2 = 1.691465 < 3.841 = \chi^2_{0.05}(1) \tag{37}$$

we do not reject the null hypothesis. That is, given the data obtained, we do not reject that the number of monthly road toll is a Poisson random variable (the second half of the definition of Poisson process). To conclude, by the evidence of Fisher's exact test of contingency tables and the Pearson's chi-squared goodness-of-fit test, Poisson process may be used to model the number of monthly road tolls as shown above.

Now we have shown that the number of monthly road tolls in the period 2008 - 2009 can be modelled as a Poisson process. In section 3.1, we develop a theoretical model which states that if Π is a Poisson process at a given time, it is again a Poisson process at some time later. We are going to perform the Pearson's chi-squared goodness-of-fit test to check if the data of monthly road tolls in 2011 - 2012 of age 18 - 25 follows a Poisson Process. Following similar procedures as we did with the data in 2008 - 2009, we first depict the observed frequency of monthly road tolls given by VicRoads in the following table:

[16]We partition in such a way that the expected frequency of each set is larger than 5, as suggested by many authors that the Pearson's chi-squared goodness-of-fit test should be used only when the expected frequency is larger than or equal to 5.

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Month	Observed Frequency O_i
January 2011	3
February 2011	2
March 2011	9
April 2011	8
May 2011	7
June 2011	9
July 2011	8
August 2011	1
September 2011	2
October 2011	5
November 2011	8
December 2011	5
January 2012	7
February 2012	6
March 2012	6
April 2012	3
May 2012	8
June 2012	3
July 2012	4
August 2012	4
September 2012	4
October 2012	5
November 2012	7
December 2012	9

Based on the data presented in the above table, the maximum likelihood estimator $\hat{\mu}$ of μ is 133/24 provided it follows a Poisson distribution. Now we test the null hypothesis, that the number of monthly road tolls is a Poisson random variable with mean 133/24 versus the alternative hypothesis that it is not a Poisson random variable.

Again, we partition the observations into sets $C_1 = \{0, 1, 2, 3, 4\}$, $C_2 = \{5, 6\}$, $C_3 = \{7, 8, 9, ...\}$. We compute the respective probabilities of each set occuring and the expected number of road tolls E_i as follow:

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-	C_1	C_2	C_3
Observed Frequency O_i	9	5	10
Probability $P(C_i)$	0.3510625	0.3284148	0.3205226
Expected Frequency E_i	8.4255	7.881955	7.692542

Using (35) and information from the table above, it follows that $\chi^2 = 1.785076$. Since we estimated $\hat{\mu} = 133/24$, χ^2 has an approximate chi-squared distribution of $3-1-1 = 1$ degrees of freedom. Now under 5% level of significance,

$$\chi^2 = 1.785076 < 3.841 = \chi^2_{0.05}(1) \tag{38}$$

we do not reject the null. That is, given the data obtained, we do not reject that $N(A_i)$ is a Poisson random variable. To conclude, by the evidence of the Pearson's chi-squared goodness-of-fit test, the monthly road tolls in 2012 follows a Poisson process. This result does not violate our theoretically developed model in section 3.1. We do not reject that the monthly road tolls in 2008 - 2009 can be modelled by a Poisson process, so do those in 2011 - 2012.

The last part of this paper regards the statistical evidence of corallary 2.11 of the marking theorem. It states the following: Given the pair (X, m_X) is a Poisson process on $S \times M$, the marks of X, denoted by m_X, is a Poisson process on the marking space M. Now we know that the monthly road tolls in 2011 - 2012 can be modelled by a Poisson process. Let the set of monthly road tolls of males in 2012 be our marking space. Here we are interested in testing whether the monthly road tolls of males of age 18 - 25 in 2011 - 2012 again follows a Poisson process as theoretically predicted by corallary 2.11. The data is depicted in the table below:

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Month	Observed Frequency O_i
January 2011	2
February 2011	2
March 2011	5
April 2011	8
May 2011	6
June 2011	9
July 2011	3
August 2011	1
September 2011	2
October 2011	5
November 2011	5
December 2011	4
January 2012	5
February 2012	5
March 2012	4
April 2012	2
May 2012	7
June 2012	2
July 2012	4
August 2012	3
September 2012	2
October 2012	3
November 2012	7
December 2012	6

As before, the maximum likelihood estimator $\hat{\mu}$ of μ is 102/24 provided it follows a Poisson distribution. Now we test the null hypothesis $N(A_i)$ is a Poisson random variable with mean 102/24 versus the alternative hypothesis that $N(A_i)$ is not a Poisson random variable. Partition the observations into sets $D_1 = \{0, 1, 2, 3\}$, $D_2 = \{4, 5\}$, $D_3 = \{6, 7, 8, 9, ...\}$. We compute the respective probabilities of each set occuring and the expected number of road tolls E_i as follow:

-	D_1	D_2	D_3
Observed Frequency O_i	10	8	6
Probability $P(D_i)$	0.3862116	0.3587275	0.2550609
Expected Frequency E_i	9.269078	8.60946	6.121462

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Again, using (35) and information from the table above, it follows that $\chi^2 = 0.103191$. Since we estimated $\hat{\mu} = 102/24$, Q_3 has an approximate chi-squared distribution of $3 - 1 - 1 = 1$ degrees of freedom. Now under 5% level of significance,

$$\chi^2 = 0.103191 < 3.841 = \chi^2_{0.05}(1) \tag{39}$$

we do not reject the null. That is, given the data obtained, we do not reject that the monthly road tolls of males is a Poisson random variable. We have shown in the beginning of this section that we do not reject the monthly road tolls being statistically independent random variables, so do those of male. Hence, by the evidence of the Pearson's chi-squared goodness-of-fit test, we do not reject that the monthly road tolls of males in 2012 again follows a Poisson process. This result does not violate our theoretical conclusion of Corollary 2.11.

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