MATHEMATICAL INVESTIGATIONS INTO RHYTHM

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1. Introduction.

1.1. **Rhythm.** Whenever we consider music, the very natural notion of a *rhythm* comes to mind. Given pitch and melody, a rhythm applies structure to a song. A rhythm is most commonly associated with a periodic pattern of note onsets and rests, and is so treated as a *cyclic* phenomenon [4, 5, 6]. This is a repeat of one or more lengths of music, or *bars*, traditionally notated on a staff as a series of notes, for example

or as commonly written down by percussionists with a series of x's and .'s

|: x . . x . . x . :|

As we frequently consider two rhythms as the same if they coincide under a cyclic permutation, it is also convenient to view a rhythm as a subset $R \subseteq \mathbb{Z}_n$ (for some n); so the above rhythm would be $\{0, 3, 5\} \subseteq \mathbb{Z}_8$. Rhythms under this definition are more precisely called *cyclic rhythms*.

Yet another convenient way to represent a rhythm $R \subseteq \mathbb{Z}_n$ is as a binary word—the *characteristic sequence* of R—with 1's denoting a beat onset, and 0's denoting a rest, for example the rhythm above has characteristic sequence

(10010010).

In building a rhythm for a melody, one desires to create a pulsed, evenly spaced feel, giving rise to this cyclic and repetitive structure. The common 4/4 rock beat found in most popular culture songs follows this simple idea, built from the trivial rhythm

(10101010).

Other rhythmic forms of music have developed lilting effects by having slightly "staggered" patterns such as the Bossa Nova

(1001001000100100),

the Samba

(1010110110 10),

and the Habenera (such as the classic example from Bizet's Carmen)

(10011010).

2. Cyclic Rhythm

2.1. **Necklaces and symmetry.** When looking at cyclic rhythms, a somewhat convenient process is to consider the rhythm as a necklace with black colourings signifying note onsets, and white colourings rests. Two rhythms are considered to be instances of the same rhythm necklace if their necklaces are equivalent up to symmetry. Recalling Burnside's Lemma, we can easily count the number of different Rhythm necklaces of a given metre with k note onsets.

Theorem 2.1. (Burnside's Lemma; *see Chapter 21 of Biggs* [2] *for instance.*) *Let a finite group G act on a finite set S; for each $g \in G$, define $\text{fix}(g)$ to be the number of elements $s \in S$ such that $g \cdot s = s$. Then the number of orbits that G induces on S is given by*

$$\frac{1}{|G|}\sum_{g\in G}\text{fix}(g).$$

We can now introduce a specific class of rhythms which we will call an Asymmetric Rhythms.

Definition 2.2. An *ℓ-Asymmetric Rhythm* is a rhythm that cannot be broken into ℓ equal duration parts such that a note onset occurs at the beginning of each part.

As an example, the *paradiddle* rhythm, a common percussionist exercise, is a 2-asymmetric rhythm:

$$\begin{array}{cc} (1001\quad 0110) & (1011\quad 0100) \\ (1101\quad 0010) & (1010\quad 0101) \end{array}$$

No matter the cyclic variation, starting the splitting the rhythm into 2 equal parts forces one part to begin with a rest.

These asymmetric rhythms were treated by Hall and Klingsberg [6], who give a proof of the following theorem.

Theorem 2.3. *The number of ℓ-asymmetric rhythm cycles of length $M = \ell n$ is given by*

$$|R_\ell^n| = \frac{1}{M}\left[\sum_{\substack{d|M \\ gcd(d,\ell)>1}}\phi(d) + \sum_{\substack{d|n \\ gcd(d,\ell)=1}}\phi(d)(\ell+1)^{n/d}\right]$$

where $\phi(d)$ denotes Euler's totient function, counting the number of integers x with $1 \le x \le d$ and x, d relatively prime.

2.2. **Euclidean Rhythm.** How does one select "natural rhythms" from the many possibilities? One reasonable consideration is to ask that onsets be distributed as evenly as possible. This constraint leads to one of the more interesting mathematical contributions to the study of rhythms. As it turns out, a number of other "naturalness" conditions lead to the same constraint, and many famous rhythms turn out to have the property. These various conditions have been studied by several authors, but the final culmination of the investigations seems to have occurred the article by Demaine, Gomez-Martin, Meijer, Rappaport, Taslakian, Toussaint, Winograd and Wood [5]. To make sense of their main result we need to introduce several further concepts, starting with the notion of a geodesic in $\mathbb{Z}_n$.

Strictly speaking, the elements of $\mathbb{Z}_n$ are equivalence classes modulo n, however we continue to abuse notation and allow any $i \in \mathbb{Z}$ to stand as a representative of its congruence class. We may now define a *standard order* on $\mathbb{Z}_n$ by $i < j$ if the smallest non-negative integer equivalent to $j \mod n$ is greater than the smallest non-negative integer equivalent to $i \mod n$. For example, under this order, $3 < -2$ in $\mathbb{Z}_8$ as -2 is equivalent to $6 \mod 8$. This enables a notion of "geodesic distance" between to elements of $\mathbb{Z}_n$.

Definition 2.4. For a given rhythm $R \subseteq \mathbb{Z}_n$, the *geodesic distance* between two onsets $i, j \in R$ is $\min\{j - i, i - j\}$, as calculated relative to the standard order on $\mathbb{Z}_n$. The *onset difference sum* of R is the value of

$$\sum_{i<j \ i,j \in R} d(i,j)$$

For a pair of positive integers $n > k$, the *Euclidean* (n,k)*-rhythm* is a k-subset of $\mathbb{Z}_n$ that maximises the onset difference sum. Demaine et al. [5] show that this is unique up to cyclic rotation.

Before we state the main theorem of this section, we need a further concept: the Bjorklund algorithm [3], which arose in an apparently unrelated context: timing systems in neutron accelerators!

Bjorklund's algorithm begins with two numbers $k < n$, and outputs a word in the alphabet $0, 1$ of length n (which we may think of as the characteristic sequence of a rhythm). We describe the algorithm by example only: in the case $k = 3$ and $n = 8$.

We begin by writing down the 3 onsets, along with the remaining $8 - 3 = 5$ rests:

$$(1)(1)(1)(0)(0)(0)(0)(0).$$

We then distribute the 0's amongst the 1's (from the right) as follows

$$(10)(10)(10)(0)(0)$$

and repeat this distribution process until only one block remains:

$$(100)(100)(10) \qquad \text{then} \qquad (10010)(100) \qquad \text{then} \qquad (10010100).$$

(In fact, up to cyclic equivalence, one can halt—and concatenate—when the next iteration will produce only a cyclic variation of the previous one).

Finally, the cutting sequence of a line through the origin, is the sequence obtained by taking a walk in the integer lattice, staying as close as possible to the line (without crossing it): horizontal steps are written as 0 and vertical steps are written as 1. Figure 1 gives an instance of a cutting sequence for a line of slope equal to the golden ratio. A line of rational slope yields a repeating pattern.

The following key result lists three of the numerous equivalent conditions established in [5, Theorem 4.1].

Theorem 2.5. *For $n > k$ positive integers, the Bjorklund algorithm outputs the Euclidean (n,k)-rhythm which coincides with the cutting sequence of a line of slope $(n-k)/k$.*

If $k|n$ then one obtains the obvious rhythm dividing n beats into k equal parts. In other instances uneven breaks are forced and the situation becomes far less trivial.

Two well known instances of uneven (but "maximally spaced") rhythms are as follows.

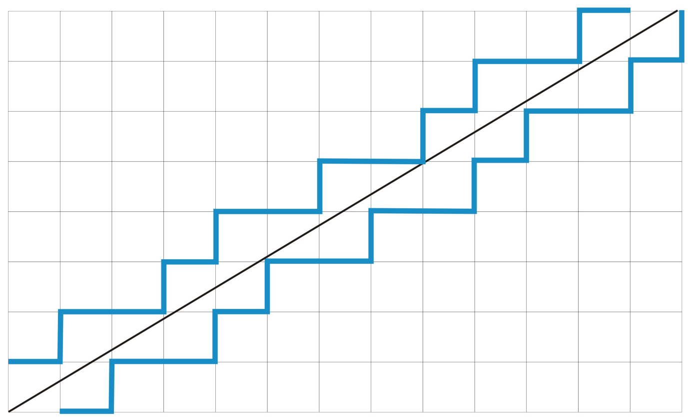

FIGURE 1. The cutting sequence of the golden ratio: 01001010010010100101 . . . .

- 10010010. One of the most famous lilting rhythms in folk, world music and modern popular music. Some examples include Metallica's Orion (guitar riff), Elvis Presley's Hound Dog (bass line), Kate Bush's Wuthering Heights (melody and piano).
- 1001001000100100. The Bossa Nova rhythm.

The article [5] lists instances of Euclidean rhythms in essentially every case with small n and k.

While Euclidean rhythms do seem to be very natural, it is important to note that not all famous uneven rhythms are Euclidean. Many of the most common Flamenco music rhythms (such as Siguiriyas and Bulerias) are based around rhythms that are *not* Euclidean; various cyclic permutations of 33222 are particularly common.

3. Acyclic Rhythm

While our definitions fixed the notion of rhythm as a finite pattern (to be repeated), there are a number of very natural infinite patterns that exhibit strong similarities to the finite case, and make for some challenging variation to the notion of rhythm. For convenience, we group these as "acyclic rhythms". In essence then, an acyclic rhythm is any subset of $\mathbb{Z}$ or $\mathbb{N}$. As before, it is usually convenient to consider these infinite "rhythms" by characteristic sequences. Thus the periodic case of even numbers in $\mathbb{N}$ is represented as 01010101 . . ., while the prime numbers would be represented as 011010100010 . . ..

We look at two instances of acyclic rhythms that exhibit very strong finiteness properties.

3.1. **Uniform recurrence.** An infinite sequence σ is *uniformly recurrent* if for every n there is an m such that every block of length n within σ occurs within each block of length m in σ (the concept comes from dynamical systems). One of the most famous instances of such a sequence is the *Thue-Morse* sequence, see Allouche and Shallit [1]; which is the infinite word generated by the morphism

$$\mu : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 10. \end{array}$$

The fixed point of this morphism is the word

$$0110100101101001100101101001 0110\ldots$$

This is an example of a fixed point rhythm, which holds the interesting property of *overlap free*, as it avoids the patterns $\alpha\beta\alpha\beta\alpha$ and $\alpha\alpha\alpha$ [7].

The rhythmic pattern generated by the Thue-Morse sequence can easily be identified as a variation of the common paradiddle rhythm, which which when analysed by subwords, is itself a paradiddle of paradiddles, revealing a fractal type nature. This ensures an interesting rhythm which continually changes and never repeats itself, whilst still remaining uniformly recurrent.

3.2. **Sturmian Words.** Another example of an acyclic rhythm can be generated from the *Fibonacci word* by the morphism

$$f: \begin{array}{c} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array}$$

The word f is said to be *balanced and aperiodic* as f does not contain a palindrome w such that both awa and bwb exist within f. This is equivalent to saying that f is *Sturmian* [7, Theorem 2.1.5].

Definition 3.1. A *Sturmian word* is an infinite word over a binary alphabet that has exactly $n+1$ factors of length n for each $n \geq 0$.

We can think of Sturmian rhythms as being the infinite, non-repeating cousins of our Euclidean Rhythms, as their construction is very much the same in the use of a cutting sequence; but instead forming a rational slope from the desired metre and number of note onsets, we use an irrational slope.

From a purely artistic view, the rhythms generated in this fashion are very interesting, which a feeling of common structure, with little complexity, whilst always remaining non-repetitive.

4. Conclusion

We can analyse rhythms mathematically for their periodic patterns, and explore what can be considered as interesting rhythms such as Asymmetric and Euclidean rhythms. These are found throughout contemporary, classical and world music; and while they are not the only apparent "natural" instances of rhythm, they contain strong and sometimes subtle mathematical structure.

The idea of Aperiodic Rhythms can be introduced such as the uniformly recurrent rhythms, or the minimum complexity Sturmian Rhythms. Future work could involve stronger characterisation of these new rhythms, as well as time spent composing music specifically built around such concepts to gain insight into the validity of how these mathematically induced ideas stand up to artistic scrutiny. Another area of further investigation would be into finding other "natural" ideas of rhythm that are also found throughout the music of today, and to further characterise them with the tools of algebra and group theory.

Working on this project has been of great benefit in gaining insight into a lighter side of the maths of the world around us, as well as giving me a great experience in research and presenting my work to my peers. It was also great to meet other people who share the same enthusiasm in mathematics.

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