Algebraic Fun in Additive Combinatorics

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Additive Combinatorics is a broad-brush term coined by Fields' medalist Terry Tao to describe a deep and burgeoning area of mathematical research that lies in the intersection of additive number theory, combinatorics, harmonic analysis, ergodic theory and abstract algebra. The aim of this field is to simply understand how the operations of addition and multiplication interact. Despite being simple, the notion of this aim is vague. I think this aim is best manifested in one of the "Holy Grail" problems of additive combinatorics: The Erdos-Turan Conjecture (1936). The conjecture states that A is an infinite subset of the natural numbers such that the sums of the reciprocals of the elements of A diverge, then A contains arithmetic progressions of arbitrary length. Green and Tao in 2004 showed this conjecture was true when the set A was the set of all prime numbers.

My project was focused on the algebraic topics of additive combinatorics. In particular I worked on problems surrounding Kemnitz's conjecture. Define a parameter $s(n,d)$ as the smallest integer s such that any sequence of s elements from ${\mathbb{Z}_n}^d$ contains a subsequence of length n whose sum is 0 in ${\mathbb{Z}_n}^d$. In order to determine $s(n,d)$ for all positive integers n and a fixed positive integer d, it suffices to determine $s(p,d)$ for all primes p. My research problem has been to consider the question:

How do we determine $s(p,d)$ for fixed values of d?

The parameter $s(n,1)$ has a fun playing card analogy. Consider a deck of n cards with each card having exactly one numerical value $0,1,\dots,n-1$ and all the values are used. A card is drawn and the value of the card is recorded. The card is replaced and the deck is shuffled. What is the least number of times this process has to occur such that we are guaranteed that we can select n of the recorded values whose sum is divisible by n?

Erdös, Ginzburg and Ziv (1961) settled this problem with the answer $s(p,1)=2p-1$. This answer came in the form of a theorem appropriately titled the EGZ theorem. The nature of the mathematics in this report has been of an additive number theoretic nature thus far, so it would not be unreasonable at this stage to ask where algebra is involved. The short answer is that the EGZ theorem relies on a classical algebraic theorem of Chevalley and Warning (1936). The Chevalley-Warning Theorem essentially asserts that given a system of homogeneous polynomial congruences over a finite field whose degree sum is smaller than the number of variables, the number of solutions is divisible by the characteristic of the finite field. Erdös, Ginzburg and Ziv use this theorem and Fermat's Little Theorem to prove the EGZ theorem.

With one dimensional case settled, Alexander Kemnitz in 1983 conjectured that $s(n,2) = 4n - 3$. This conjecture was open for twenty years until C.Reiher (an undergraduate!) and C. de Foire (a high school student!) independently solved the conjecture. Both of their proofs have the same crucial argument but only differ in the preliminaries. A crucial lemma used is the Alon and Dubiner lemma that says that the existence of a zero sum subsequence of length $3p$ implies there is one of length p, where p is an odd prime. In particular C.Reiher makes use of the Chevalley-Warning Theorem to derive congruences involving terms that count the number of zero sum subsequences of prescribed sizes. Combining this with ingenious binomial coefficient and combinatorial arguments he solved the conjecture.

So this brings us to the still open case of determining $s(p,3)$ for all odd primes p. It has been conjectured (Sun 2009?) that $s(p,3) = 9p - 8$ for all odd primes p. It is known that $s(p,3) \geq 9p - 8$ and that equality holds in this bound when $p = 3$. The bulk of my project has been trying to prove equality in the aforementioned lower bound for all $p > 3$. This has been challenging, and the problem has captivated me to the point where I plan to continue to work on my project after my Australian Mathematical Sciences Institute Vacation Research Scholarship has finished.

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