Sheaves and Algebraic Curves

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Algebraic geometry is a vast and beautiful branch of mathematics. In the beginning, algebraic geometers studied the loci of polynomial equations in both affine and projective space. Modern algebraic geometers study objects called schemes. In order to understand schemes, one needs to understand sheaf theory.

Let X be a topological space. A presheaf F over X is given by the following data: for each open set U in X, an abelian group F(U) and for each inclusion of open sets $V \subseteq U$, a homomorphism from F(U) to F(V). The elements of F(U) are called sections over U and the homomorphisms are called restriction maps. We also require that restriction from F(U) to F(U) is the identity and restriction form F(U) to F(V) and then F(V) to F(W) is the same as restriction from F(U) to F(W). A sheaf is a presheaf such that all locally zero sections are zero and given a collection of compatible sections, they glue together. An example of a sheaf: let F(U) be the continuous functions from U to the complex numbers, and the restriction maps normal function restriction.

At first glance, sheaf theory is just a language used to generalize functions on a topological space, for example differentiable functions on a manifold. It becomes much more when one realizes that sheaves over a topological space form an abelian category. An abelian category is one of the most general settings in which one can do homological algebra.

If our topological space X is equipped with a sheaf of rings, we can define the abelian category of coherent sheaves on X. The locally rank 1 sheaves in this category form an abelian group (multiplication is tensor product and the inverse of a locally rank 1 sheaf is its dual). This group is called the picard group.

During the summer I learnt about sheafification, a process which deforms a presheaf into a sheaf by identifying locally zero sections with zero and adding in extra sections so that the gluing axiom is satisfied. Sheafification plays a major role in proving that sheaves form an abelian category. It is also important for constructing the tensor product of two sheaves.

I also learnt about algebraic curves over the complex numbers. I learnt that the picard group of an algebraic curve is isomorphic to divisors modulo linear equivalence, line bundles and the first cohomology group of a special sheaf on the algebraic curve. I spent some time learning about elliptic curves (non singular projective algebraic curves of genus 1). It is a beautiful fact that when X is an elliptic curve, there is a set bijection between the picard group and X. This bijction endows X with a group structure.

I would like to thank my supervisor Dr Emma Carberry for teaching and supporting me over the summer. I would like to thank AMSI, CSIRO and the University of Sydney for their support.

Daniel Barter received a 2010/11 AMSI Vacation Research Scholarship