Estimation for ACD and log-ACD models

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1 Introduction

The Autoregressive Conditional Duration (ACD) model (developed by Engle and Russell (1998)) and its variations are used for analysis of data which arrive at irregular time intervals. In particular, they have been used heavily in finance to model irregular time intervals between trades or price changes of stocks. Many financial theories focus on transaction-by-transaction basis, so the timing of these transactions can be essential for understanding the economics. An important application of the model is to measure and forecast the duration of transaction arrivals, which is essentially the instantaneous quantity of transactions. The model parameterises the conditional duration as a function of the time between past events. It is the dependence of the conditional duration on past time intervals, which suggests that the model be called the autoregressive conditional duration (ACD) model.

Section 2 introduces the ACD model, and discusses some properties and constraints, while Section 3 introduces the log-ACD model, which is a more flexible variate of the ACD model. Section 4 describes four conditional probability distributions (the exponential, Weibull, generalised gamma and truncated skewed student-t distributions). In Section 6, these distributions are used to simulate data using the ACD and log-ACD models. The properties of the maximum likelihood estimator of the parameters are also discussed in Section 6. Finally, the models are fitted to a dataset containing transaction durations of IBM stock, and these models are compared in Section 7.

2 The ACD Model

The Autoregressive Conditional Duration (ACD) Model was developed by Engle and Russell (1998) to model irregular time intervals between trades. We let $x_i = t_i - t_{i-1}$ denote the duration between the $(i-1)$th and the ith trade. We further let $\psi_i = E(x_i|F_{i-1})$ be the conditional expectation of x_i given F_{i-1}, the information set available at the $(i-1)th$ trade, which contains at least $\tilde{x}_{i-1}$ and $\tilde{\psi}_{i-1}$, where $\tilde{x}_{i-1}$ denotes x_{i-1} and its past values, and likewise for $\tilde{\psi}_{i-1}$.
The ACD model is defined by:

$$x_i = \psi_i \epsilon_i \tag{1}$$

where ϵ_i is a sequence of independent and identically distributed (i.i.d) non-negative random variables such that $E(\epsilon_i) = 1$. ϵ_i can be chosen to follow many distributions such as the exponential or Weibull distributions (to be discussed in Section 3). Further ψ_i has the form:

$$\psi_i = \omega + \sum_{j=1}^{p} \alpha_j x_{i-j} + \sum_{j=1}^{q} \beta_j \psi_{i-j} \tag{2}$$

This model is called an ACD(p,q) model.
Note that $\eta_i = x_i - \psi_i$ is a martingale difference sequence, that is $E(\eta_i|F_{i-1}) = 0$. Substituting into equation (2), and assuming that $\alpha_j = 0$ for $j > p$ and $\beta_j = 0$ for $j > q$ we get

$$x_i = \omega + \sum_{j=1}^{max(p,q)} (\alpha_j + \beta_j) x_{i-j} - \sum_{j=1}^{q} \beta_j \eta_{i-j} + \eta_j \tag{3}$$

This model is now in the form of an ARMA process with non-Gaussian errors/innovations. Now, taking expectation on both sides of equation (3) (note that $E(\eta_i) = E(E(\eta_i|F_{i-1})) = E(0) = 0$), and assuming that the model is weak stationary (that is, has a constant mean and that the covariance function depends only on the difference between t_1 and t_2, and not the actual values of t_1 and t_2), we get:

$$E(x_i) = \frac{\omega}{1 - \sum_{j=1}^{max(p,q)} (\alpha_j + \beta_j)} \tag{4}$$

Hence, to satisfy the requirement that the expected duration is positive, we require that $\omega > 0$ and $1 > \sum_j (\alpha_j + \beta_j)$. Moreover, to ensure positivity of the conditional durations, we also require that $\beta \geq 0$ and $\alpha \geq 0$.

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3 The Logarithmic-ACD Model

We now introduce the logarithmic version of the ACD model. This model is considered more flexible than the original ACD model as we do not require the coefficients to be positive. Again, we let x_i denote the duration between the $(i-1)$th trade and the ith trade. The log-ACD model can be expressed as:

$$x_i = \exp(\phi_i)\epsilon_i \tag{5}$$

$$\phi_i = \omega + \sum_{j=1}^{p} \alpha_j \text{log}(x_{i-j}) + \sum_{j=1}^{q} \beta_j \phi_{i-j}, \tag{6}$$

where $\exp(\phi_i)$ is the conditional expectation of the duration at observation i i.e. $E(x_i|F_{i-1}) = \exp(\phi_i)$, and ϵ_i is a sequence of independent and identically distributed (i.i.d) non-negative random variables such that $E(\epsilon_i) = 1$.
We can rewrite (5) as:

$$\text{log}x_i = \phi_i + \mu_i \tag{7}$$

where $\mu_i = \text{log}\epsilon_i$.

4 Conditional Distributions

We now consider several distributions for ϵ_i, namely the exponential, Weibull, generalised gamma and truncated skewed student-t distributions. The generalised gamma (η, λ, κ) distribution has a shape parameter η, scale parameter λ and a third parameter κ. The Weibull (λ, η) distribution is a special case of the generalised gamma distribution with $\kappa = 1$, and the exponential distribution is a further special case with $\kappa = \eta = 1$. We also consider the truncated skewed student-t distribution. Recall that for both the ACD and the log-ACD models, we require that $E(\epsilon_i) = 1$, thus we consider the standardised distributions.
The log-likelihood function is defined by

$$l(\theta|\mathbf{x}) = \sum_{\mathbf{i=1}}^{\mathbf{T}} \text{log}\mathbf{f}(\mathbf{x_i}; \theta)$$

where θ is a vector of parameters. Table 1 shows the standardised pdfs for each of these distributions, while Table 2 shows the log-likelihood functions.

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	Standardised pdf
Exponential	e^{-x}
Weibull	$\alpha\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^{\alpha}y^{\alpha-1}\exp\left(-\left[\Gamma\left(1+\frac{1}{\alpha}\right)y\right]^{\alpha}\right)$
Gen. Gamma	$\frac{\alpha y^{\kappa\alpha-1}}{\lambda^{\kappa\alpha}\Gamma(\kappa)}\exp\left[-\left(\frac{y}{\lambda}\right)^{\alpha}\right]$
T.S. Student-t	$\frac{2}{1+\lambda}c\left[1+\frac{1}{\nu-2}\left(\frac{y_i}{1+\lambda}\right)^2\right]^{-(\nu+1)/2}$

Table 1: Standardised probability density formula

	log-likelihood $l(\boldsymbol{\beta}\mid\mathbf{x})$
Exponential	$\sum_{i=i_0+1}^{T}-\log(\psi_i)-\frac{x_i}{\psi_i}$
Weibull	$\sum_{i=i_0+1}^{T}\alpha\ln\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]+\ln\left(\frac{\alpha}{x_i}\right)+\alpha\ln\left(\frac{x_i}{\psi_i}\right)-\left(\frac{\Gamma(1+1/\alpha)x_i}{\psi_i}\right)^{\alpha}$
Gen. Gamma	$\sum_{i=i_0+1}^{T}\ln\left(\frac{\alpha}{\Gamma(\kappa)}\right)+(\kappa\alpha-1)\ln(x_i)-\kappa\alpha\ln(\lambda\psi_i)-\left(\frac{x_i}{\lambda\psi_i}\right)^{\alpha}$
T.S. Student-t	$\sum_i-\log(\psi_i)+2\log(2c)+\log(\frac{\nu-2}{\nu-1})-\left(\frac{\nu+1}{2}\right)\log\left(1+\frac{4c^2(\nu-2)}{(\nu-1)^2}\left(\frac{x_i}{\psi_i}\right)^2\right)$

Table 2: Log-likelihood formula for the ACD(p,q) model

5 Markov Chain Monte Carlo (MCMC) Method

The MCMC method for sampling from a probability distribution, $P(x)$, involves constructing a Markov chain whose equilibrium distribution is the (desired) target probability distribution. The state of the chain after a large number of steps is then used as a sample of the target distribution. .
We now describe one particular MCMC method, the Metropolis−Hastings algorithm.

- First, choose an arbitrary probability density $Q(x'|x_t)$, called the *proposal distribution* (We chose the proposal distribution to be the standard normal distribution), which suggests a new sample value x' given a sample value x_t. This proposal density is symmetric, that is, $Q(x'|x_t) = Q(x_t|x')$.

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- Choose an arbitrary point x_0 as the first sample.
- Then, to generate a new sample point x_{t+1} given the most recent sample point x_t, we perform the following algorithm:
  1. Generate a proposed new sample value x' from the proposal distribution $Q(x'|x_t)$
  2. Calculate the acceptance ratio, $\alpha = \frac{P(x')}{P(x_t)}$, then let $r = \min(1, \alpha)$
  3. Accept x' with probability r. That is, pick a uniformly distributed random number u between 0 and 1, and if $u \leq r$, set $x_{t+1} = x'$, else set $x_{t+1} = x_t$

Note that the acceptance ratio α indicates how probable the new proposed sample is with respect to the current sample, according to the distribution $P(x)$. As it takes many steps for the Markov Chain to reach its equilibrium distribution, we introduce a 'burn-in' period, where the first M (we take $M = 5000$) samples are thrown away.

6 Simulation and Estimation

We use a Markov Chain Monte Carlo Method (MCMC) to simulate data from the desired distribution (standardised to have unit mean) of ϵ_i. We generated 2000 series, each consisting of 500 observations firstly from the ACD(1,1) model given by:

$$x_i = \psi_i \epsilon_i, \qquad \psi_i = 0.3 + 0.2x_{i-1} + 0.7\psi_{i-1} \tag{8}$$

and then the log-ACD(1,1) model given by:

$$x_i = \exp(\phi_i)\epsilon_i, \qquad \phi_i = 0.3 + 0.2x_{i-1} + 0.7\psi_{i-1} \tag{9}$$

using the exponential, Weibull, generalised gamma and truncated skewed student-t distributions for ϵ_i. Figure 1 shows time series plots for each of the ACD and log-ACD models.

A Maximum Likelihood estimation (MLE) procedure was used to estimate the parameters of the simulated ACD and log-ACD models. The MLE is defined by:

$$\hat{\theta}_{\text{mle}} = \arg\max_{\theta} l(\theta|\mathbf{x})$$

where θ is a vector of parameters of interest, and $l(\theta|\mathbf{x})$ is the associated log-likelihood function.

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The average of the 2000 estimates for each parameter are given in Table 3. Histograms of these estimates for each parameter are given in Figures 2, 3 and 4. These histograms imply that the maximum likelihood estimates follow a roughly normal distribution.

Parameter	ω	α_1	β_1
True	0.3	0.2	0.7
EACD(1,1)	0.314 (0.037)	0.223 (0.032)	0.724 (0.037)
SktACD(1,1)	0.311 (0.038)	0.225 (0.031)	0.727 (0.036)
WACD(1,1)	0.310 (0.037)	0.229 (0.029)	0.732 (0.033)
GACD(1,1)	0.317 (0.038)	0.218 (0.035)	0.718 (0.039)
log-EACD(1,1)	0.319 (0.023)	0.239 (0.030)	0.740 (0.033)
log-SktACD(1,1)	0.314 (0.022)	0.239 (0.032)	0.738 (0.034)
log-WACD(1,1)	0.316 (0.020)	0.252 (0.032)	0.722 (0.034)
log-GACD(1,1)	0.320 (0.038)	0.235 (0.030)	0.746 (0.034)

Table 3: Mean MLE for 2000 simulated ACD and log-ACD, each with 500 observations

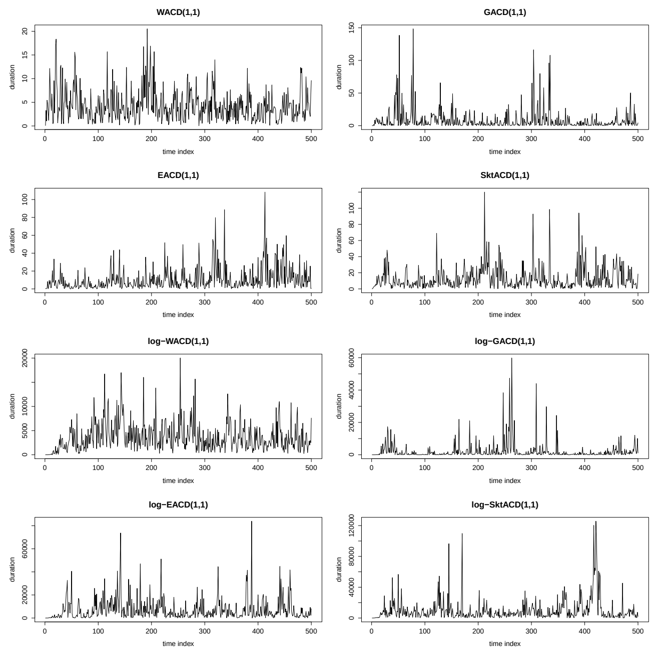

Figure 1: Simulated ACD(1,1) and log-ACD(1,1) series

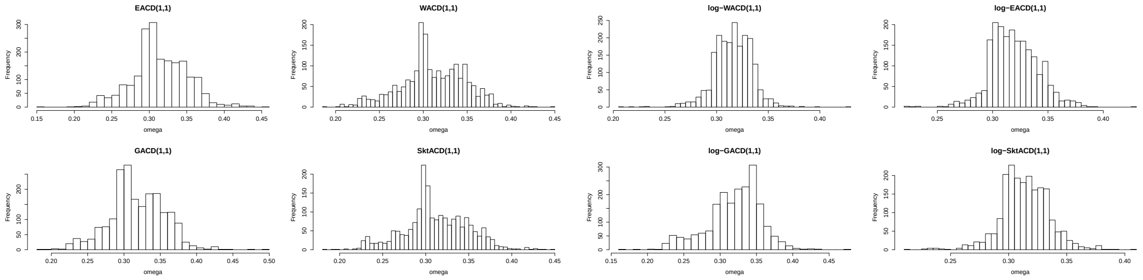

Figure 2: Histogram of the ML estimates of ω generated by 2000 iterations of each ACD(1,1) and log-ACD(1,1) series given by (15) and (16), each of length 500

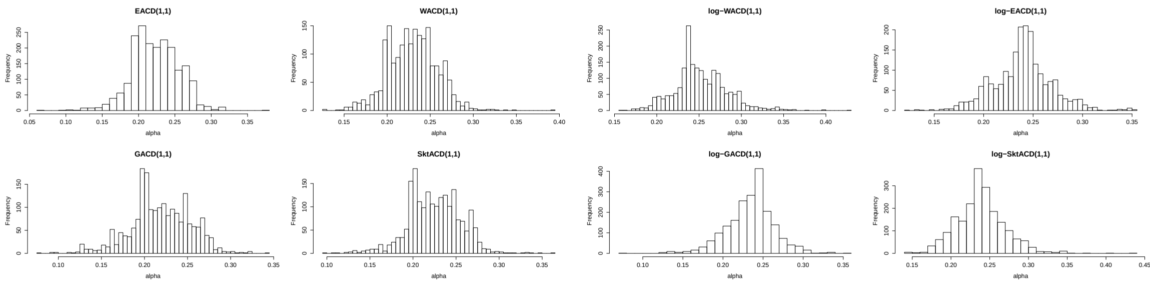

Figure 3: Histogram of the ML estimates of α_1 generated by 2000 iterations of each ACD(1,1) and log-ACD(1,1) series given by (15) and (16), each of length 500

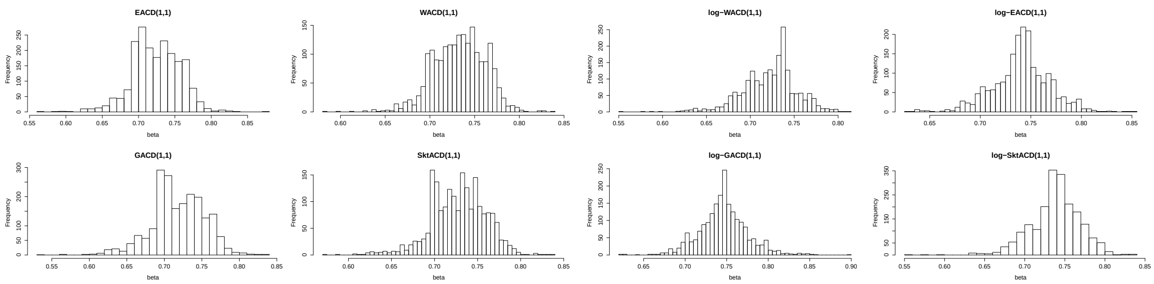

Figure 4: Histogram of the ML estimates of β_1 generated by 2000 iterations of each ACD(1,1) and log-ACD(1,1) series given by (15) and (16), each of length 500

7 IBM dataset example

We consider the transaction durations of IBM stock on five consecutive trading days from November 1 to November 7 1990. We have 3534 observations. A time series plot of the series is given in Figure 6.

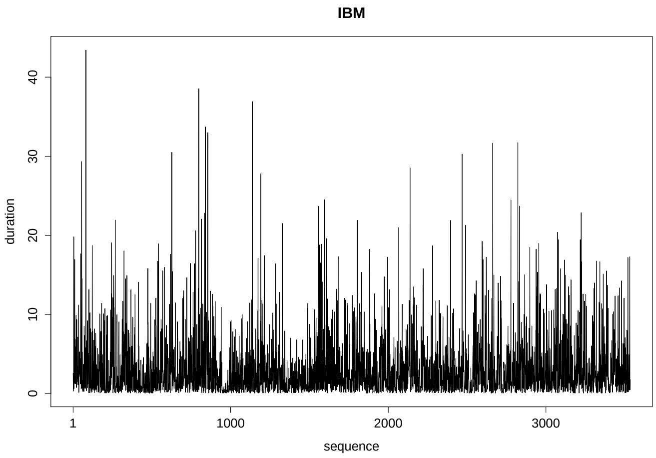

Figure 5: Time series plot of IBM stock traded in the first five trading days of November 1990

We fitted each ACD(1,1) and log-ACD(1,1) model described in the previous section to the IBM data, and the results are given in Table 2. For each model the first 2534 observations x_i for $i = 1, \ldots, 2534$ were used to generate the model from which 1000 forecasted values $\hat{x}_i$ for $i = 2535, \ldots, 3534$. The Mean Squared Error (MSE) for each model was then calculated using the 1000 forecasted values and comparing them to the final 1000 real observations. This procedure is called the in-sample out-of-sample error, and is calculated by:

$$MSE = \frac{1}{n} \sum_i (\hat{x}_i - x_i)^2$$

From the MSE values in Table 4., it appears that the log-WACD model is the best fit for the data, and is given by:

$$x_i = \exp(\phi_i)\epsilon_i \qquad \phi_i = 0.111 + 0.068 \log(x_i) + 0.874\phi_{i-1}$$

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	ω	α_1	β_1	MSE
EACD(1,1)	0.189 (0.055)	0.078 (0.012)	0.864 (0.024)	32.432
WACD(1,1)	0.166 (0.055)	0.075 (0.012)	0.873 (0.025)	33.026
GACD(1,1)	0.114 (0.039)	0.071 (0.012)	0.896 (0.018)	40.216
SktACD(1,1)	0.147 (0.051)	0.074 (0.012)	0.886 (0.022)	48.278
log-EACD(1,1)	0.117 (0.026)	0.070 (0.010)	0.869 (0.026)	30.508
log-WACD(1,1)	0.111 (0.026)	0.068 (0.011)	0.874 (0.027)	29.838
log-GACD(1,1)	0.122 (0.032)	0.073 (0.013)	0.862 (0.033)	32.552
log-SktACD(1,1)	0.122 (0.022)	0.065 (0.009)	0.877 (0.020)	33.467

Table 4: Maximum likelihood estimates for the parameters of various ACD(1,1) models when fitted to the IBM dataset. Associated standard deviations are in parentheses

8 Conclusion

The aim of this project was to study the ACD and log-ACD models under various probability distributions. Through extensive simulation, it is clear that the log-ACD model is much easier to deal with due to increased flexibility in the parameters. Moreover the maximum likelihood estimator obtains quite good estimates of the parameters, however the MLE does appear to overestimate the parameters, for reasons unknown at this point.
Through analysis of the IBM dataset, it was found that the model which best fitted the data was the WACD(1,1) model (The ACD model using a Weibull distribution). This could be used to predict future transaction durations for IBM.

9 References

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