Geometric Langlands Program

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Abstract

This report provides a brief overview of the objects involved in the geometric Langlands conjecture. It provides a summary of my research project over the course of the AMSI Vacation Scholarship in 2013/14. The final section looks at how this has developed into my Honours project for the remainder of this year.

1 Introduction

The global geometric Langlands correspondence (in various formulations) is a conjectural link between ${}^LG$-local systems on a Riemann surface X (where ${}^LG$ is the Langlands dual group — a group defined in terms of G as defined in e.g. [Bou08, Fre07a], and certain D-modules on the moduli stack Bun_G of principal G-bundles on X for a general reductive group G. There are multiple conjectured versions of this correspondence, for example the categorical geometric Langlands conjecture, which conjectures this link is a functor between derived categories of these objects, however this will not be looked at in this report.

There is also a local correspondence in the geometric case. This conjectures a link between ${}^LG$-local systems on discs around a point $x \in X$ to subcategories of the category of $\widehat{\mathfrak{g}_\kappa}$ modules, where $\widehat{\mathfrak{g}_\kappa}$ is the affine Kac-Moody algebra of $\mathfrak{g}$ (a central extension of $L\mathfrak{g}$). This concerns the local information about the global objects in the global correspondence, as the discs are defined so that the functions on them correspond to the formal Laurent series around a point, and the Lie algebras are linked to the D-modules by the Beilinson–Bernstein correspondence which is mentioned in section 3.

The geometric Langlands correspondence is a geometric analogue of the arithmetic global Langlands correspondence which conjectures a link between the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of an algebraic number field F and certain automorphic representations. When rewritten in the case of other global fields - that is function fields over an algebraic curve over a finite field it can be reformulated geometrically. This geometric reformulation also makes sense for curves over $\mathbb{C}$ and this gives the geometric case [Fre07b]. Currently the non-categorical version is proven only for the case $G = GL(n, \mathbb{C})$.

2 Geometric Abelian Class Field Theory

This section provides a brief description of geometric class field theory based on that in [Lau02].

The *Jacobian* $J(X)$ of a compact, connected, Riemann surface X, is the moduli space of line bundles of *degree* zero. For a line bundle L on a Riemann surface X and any non-zero meromorphic section $s : X \to L$, the degree of L is the number of poles of s minus the number of zeros of s (counting multiplicities). The trivial bundle is hence clearly an example of a line bundle of degree zero, as can be seen by taking a non-zero constant section. For a Riemann surface X the Jacobian is a complex torus of dimension equal to the genus g of X, so $J(X) = \mathbb{C}^g/\Lambda$, for a lattice Λ which can be calculated using topological properties of X.

There is a product m on the Jacobian which corresponds to taking the tensor product of line bundles. This is the product on the *Picard variety*, a variety which parametrises all line bundles on X.

We can also use the topological properties of X to give a map $\phi : X \to J(X)$ which maps a chosen point x_0 to the group identity (the point zero in the torus).

The following theorem gives abelian class field theory. A rank one local system refers to a line bundle with a flat connection, and local systems are described further in section 3. Consider the standard projections $\pi_1, \pi_2 : X \times X \to X$, then for line bundles E_1 and E_2 over X we define their exterior tensor product $E_1 \boxtimes E_2$ over $X \times X$ to be the vector bundle $\pi_1^* E_1 \otimes \pi_2^* E_2$.

Theorem 2.1. *For each rank one local system E on X there exists a unique rank one local system Aut_E on $J(X)$, with $\phi^* Aut_E = E$ and $m^* Aut_E \cong Aut_E \boxtimes Aut_E$.*

We can now outline how these objects correspond to the geometric Langlands correspondence for an arbitrary group $GL(n, \mathbb{C})$ for $n = 1$. For other values of n we change the rank one local system E to a rank n local system. However the rank one

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local system on $J(X)$ changes more drastically. It is replaced by a D-module on the moduli space of vector bundles of rank n on X which transforms in certain ways under a class of functors known as the Hecke functors. This raises the question of why in the $n = 1$ case we only define Aut_E on $J(X)$ rather than on the Picard variety $Pic(X)$ which is the moduli space of rank one vector bundles. The reason for this is that in the one dimensional case the way that that Aut_E transforms under Hecke functors allows the construction of a D-module on the components of $Pic(X)$ which correspond to line bundles of degrees other than zero.

3 D-modules and Local Systems

We now introduce D-modules. A D-module is a type of *sheaf*, we first define a sheaf;

Definition 3.1. *A sheaf $\mathcal{F}$ on a topological space Y is a functor that associates to each open set $U \subset Y$ an object $\mathcal{F}(U)$ of some category $\mathcal{C}$, and to each inclusion $U \subset V$ of open sets a restriction morphism $r_{U,V} : \mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$, such that the following conditions are satisfied:*

1. $r_{U,U} = Id_U$.
2. *For* $U \subset V \subset W$, $r_{U,V} r_{V,W} = r_{U,W}$.
3. *For an open cover* $\{U_i\}_{i \in I}$ *of a set* U *if for* $s, t \in \mathcal{F}(U)$, $r_{U_i,U} s = r_{U_i,U} t \ \forall i \in I$ *then* $s = t$.
4. *For an open cover as in the above condition, with a element* $s_i \in \mathcal{F}(U_i) \ \forall i$ *such that* $\forall i, j \in I \ r_{U_i \cap U_j, U_i} s_i = r_{U_i \cap U_j, U_j} s_j$ *then there is an element* $s \in U$ *such that* $r_{U_i,U} s = s_i$ *for all* i.

An example of a sheaf is given by the sheaf of continuous functions on any topological space.

An example of a sheaf of rings (that is to say that $\mathcal{F}$ is a functor to the category of rings) is the sheaf of differential operators, $\mathcal{D}$, on a manifold X.

A *D-module* $\mathcal{F}$ (over a manifold X) is a sheaf of modules over the sheaf of differential operators on X, that is to say $\mathcal{F}(U)$ is a module over $\mathcal{D}(U)$ for every open set $U \subset X$. In the case here the topological space in question is a stack, the way that the topology and the sheaf of differential operators can be defined on a stack is given in [Góm01].

These D-modules are the analogues of automorphic representations in the arithmetic case [Fre07b]. Another link to representations is given by Beilinson–Bernstein

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localisation. Beilinson–Bernstein localisation gives a link between D-modules and the representations of certain Lie algebras.

Let G be a reductive algebraic group, a *Borel subgroup* B is a subgroup that is maximal in the family of connected, solvable, closed subgroups of G. The *flag variety* of an algebraic group G is the algebraic variety $X = G/B$. The action of G on X gives an infinitesimal action of $\mathfrak{g}$ on the functions on any subset $U \subset X$, i.e. an action on the space of sections of the trivial line bundle $X \times \mathbb{C}$ over X. This maps $\mathfrak{g}$ to the tangent space of X. From this we can induce a map from the universal enveloping algebra of $\mathfrak{g}$ to the sheaf of sections of X. Pulling back this map gives a map

$$\Lambda^* : \mathrm{Mod}(D_X) \to \mathrm{Mod}(U(\mathfrak{g})),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ [Bou08].

The Beilinson–Bernstein theorem states that there is also a map

$$\Delta : \mathrm{Mod}(U(\mathfrak{g})) \to \mathrm{Mod}(D_X).$$

Furthermore we can look at D-modules not over sections of the trivial line bundle, ie. functions $X \to \mathbb{C}$ but also over other line bundles, and the topological properties of the line bundle can be used to derive properties of the module over $U(\mathfrak{g})$ [HT08]. The functor given by Beilinson-Bernstein can be used to relate the subcategories of the category of $\widehat{\mathfrak{g}}_\kappa$-modules to the D-modules on Bun_G, as is outlined in [Fre07a].

The Hecke functors act on the D-modules on Bun_G in such a way that we can introduce the notion of eigen-D-modules, with the eigenvalue being a ${}^LG$-local system on X. Hence we can link each Hecke eigen-D-module to a local system - specifically the local system that is its eigenvalue. The geometric Langlands correspondence conjectures that we can also link ${}^LG$-local systems E to a Hecke eigen-D-module on Bun_G with eigenvalue E [Fre07b].

We now introduce the local systems which the geometric langlands correspondence links certain D-modules to. The local systems are related to homomorphisms of the fundamental homotopy group of the curve X to the Langlands dual ${}^LG$. We will now illustrate how this works for the case where $G =^L G = GL(n)$.

A *local system* on a Riemann surface X is equivalent to a vector bundle E on X, with a *flat connection*, that is a connection ∇_α such that $[\nabla_\alpha, \nabla_\beta] = \nabla_{[\alpha,\beta]}$ is satisfied, where $\alpha, \beta \in \Gamma(TX)$.

A connection allows us to gain a vector in E_y associated to a given vector in E_x given a path between the points $x, y \in X$. In the case that $x = y$ we are hence associating a path starting and beginning at x to a transformation of the vector space E_x. This transformation is linear, and in the case when ∇ is a flat connection has the property

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that homotopy equivalent paths will give the same linear transformation of E_x. Thus a local system corresponds to a representation $\pi_1(X,x) \to GL(E_x) = GL(n,\mathbb{C})$.

4 Links to Electromagnetic Duality

This section outlines the recent links found between electromagnetic duality and the geometric Langlands program in [KW06]. For more details on these links see e.g. [Fre09, KW06].

The Maxwell equations for electromagnetism, in a vacuum (under suitable choice of constants made possible by scaling) are given by

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{E} = \tfrac{\partial \mathbf{E}}{\partial t} \tag{2}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively. These equations are invariant under the transformation $(\mathbf{E},\mathbf{B}) \to (\mathbf{B},-\mathbf{E})$. There are several other versions of electromagnetic duality, in which a theory is invariant under this switching of electric and magnetic fields, for example in the $U(1)$ gauge theory (without sources) describing electromagnetism.

However when describing physical theories we often need to use a *Yang-Mills Theory*, a gauge theory in which the gauge group is non-abelian, for example this is the case for the standard model which has the non-abelian gauge group $U(1)\times SU(2)\times SU(3)$. In this case the conjectured Montonen–Olive duality [MO77] gives a duality between the theory with gauge group G, and the theory with the gauge group ${}^LG$ with the electric and magnetic fields switched, and certain constants modified.

The fact that electromagnetic duality links a group G with its Langlands dual group ${}^LG$ suggested that electromagnetic duality might be linked to the Langlands program. In [KW06] such a link is described, and this is outlined in the following paragraphs.

A *topological field theory* is a field theory in which the observables are invariant under deformations of the metric [Loz99]. We can gain a four dimensional topological field theory from $\mathcal{N}=4, d=4$ supersymmetric Yang-Mills theory by a process known as topological twisting, in which a supersymmetric operator Q is formed such that $Q^2=0$, and the state space of the resulting field theory is given by the cohomology of Q, i.e.

$$\frac{\mathrm{Ker}(Q)}{\mathrm{Im}(Q)}.$$

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For this reason this type of topological field theory is also a cohomological field theory.

In the case that the base space is the product of two manifolds (each of dimension 2), ie $X = \Sigma \times C$, we can shrink C, and take the limit as the area of C tends to zero to gain a two dimensional topological quantum field theory.

This two dimensional theory is a type of sigma model called an A-model - its fields correspond to maps from Σ to $\mathcal{M}_H(G, C)$, the Hitchin Moduli space. The Hitchin moduli space is the space of solutions (on C) to the Yang-Mills equations reduced to two dimensions, however also have several other important descriptions linking it to G-bundles on C, for an introduction to these see Garcia-Prada's appendix to [Wel08].

If we now consider the topological field theory we get by applying the same process to the theory that is dual to the original Yang-Mills theory started with under the Montonen–Olive duality we get a B-model - its fields correspond to maps from Σ to $\mathcal{M}_H({}^LG, C)$. The duality between the original two Yang-Mills theories implies a duality between certain *branes* in the A-model and the B-model, these branes correspond to boundary conditions on the maps $\Sigma \to \mathcal{M}_H(G, C)$ and $\Sigma \to \mathcal{M}_H({}^LG, C)$. For example in the first case a zerobrane is a point $p \in \mathcal{M}_H(G, C)$, and applying this boundary condition to $\gamma \subset \partial\Sigma$ means requiring that a map maps γ to the point p. It is suggested that the A-branes correspond to D-modules, and this allows the link between A-branes and B-branes given by Montonen-Olive duality to be seen as geometric Langlands duality, albeit written in a different form. With regards to the duality between the A and B-models there are still some open questions on how structures, such as fermion bundles, on the manifold C behave with respect to this duality.

In [KW06] Montonen–Olive duality is applied to manifolds of the form $X = \Sigma \times C$, it would be interesting to see whether variations on this form of manifold, such as non-trivial C-bundles over Σ, could also give dualities related to Langlands program. Also in [Wit09] Witten outlines the idea of using a six dimensional conformal field theory to give results about the Yang-Mills theories, and hence about geometric Langlands. Given that two-dimensional conformal field theory has been used in the local Langlands correspondence, see e.g. [Fre07b], it seems plausible that this could be linked to the six dimensional conformal field theory Witten discusses.

5 Further Work

This year I will be studying towards an Honours degree at ANU, in which I will be studying the paper [KW06] in greater depth to look further into the link between gauge theory and geometric Langlands. I aim to look further into some of the questions raised in the above section.

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