Can one hear the shape of a drum?

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"Hearing a drum" means hearing all of the sounds that vibrations of the drumhead of a drum can make when struck. There is an uncountable infinity of different vibrations that any given drumhead can generate. Fortunately, it is possible to decompose every vibration into simpler constituent oscillations, called "pure tones", of which the original vibration is a superposition. Moreover, the set of pure tones of a drum is countably infinite. To hear a drum is therefore a simple matter of knowing all the pure tones of a drum.

This can be made precise mathematically: drumheads are modelled by bounded planar domains, the vibrations obey the two-dimensional wave equation, and pure tones and their frequencies of oscillation correspond to eigenfunctions and eigenvalues of the two-dimensional Laplacian operator respectively.

An interesting problem for mathematicians and musicians alike is whether one can infer all topological and geometric properties of a drum merely from hearing it. Put another way, if we know all the eigenvalues of the Laplacian operator acting on the drumhead, can we recreate the drum? This question was posed in its mathematical form in a famous paper by *Mark Kac* in 1966. This paper was the central focus of my project with *Dr. Andrew Hassell* of the ANU: to read it thoroughly and understand its mathematical arguments, intricacies, and beauty.

Kac proved rigorously in his paper that we are able to hear the area and perimeter of drumheads. That is, knowing its eigenvalues implies knowledge of the area and circumference of the drumhead. He went on to conjecture that one can also hear the Euler characteristic (the number of holes) of drumheads. This was later proved to be true in 1967 by H.P. McKean and I.M. Singer.

It was ultimately shown in subsequent papers in the literature that the answer to Kac's original question is negative. We cannot deduce all topological and geometric properties of a drum from its pure tones. A counterexample, a pair of simply connected planar domains with polygonal boundaries that had identical sets of eigenvalues, was provided by C. Gordon, D. Webb, and S. Wolpert in 1992.

A final third of my project involved using techniques such as boundary blowups and Richard Melrose's b-calculus to developed an asymptotic expansion of the heat kernel on planar domains. As it turns out, an asymptotic expansion of the heat kernel would reveal more geometric and topological properties that can be heard of a drum.

My project would not have been possible without the generous financial support of AMSI and the ANU, which made it possible for me to live and work in Canberra during the summer. My greatest gratitude is to Andrew Hassell for his unfailing guidance throughout this project.

Xingyu received an AMSI Vacation Scholarship in December 2010 See: www.amsi.org.au/vs10.php