Expansions of Zeta

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The Riemann zeta function was first studied by Euler in the eighteenth century, and has since become the holy grail of number theory. Riemann showed that the zeroes of the zeta function are linked with the distribution of the prime numbers in the naturals, and his still unproven 1859 hypothesis on the locations of the zeroes is now one of the six remaining Millennium problems.

The term "zeta function" has since come to denote any function analogous to the zeta function, usually via a similar definition. My research was focused on the p-adic zeta functions (often considered as branches of a single "function") defined by Kubota and Leopoldt in 1964, which are continuous p-adic functions obtained by interpolating certain special values (the Bernoulli Numbers) of the classical Riemann zeta function. The objective of the project was to gain an understanding of the mathematics involved, and to use series expansions of the zeta functions to investigate their structure.

My starting point was to learn the basics of algebraic number theory, and then to understand the construction of the p-adic fields (where p is a fixed odd prime) $\mathbf{Q}_p$ and $\mathbf{C}_p$ and the p-adic zeta functions. I also familiarised myself with the programming environment PARI/GP, which I used throughout the project for all my numerical computations.

The classical zeta function has known values at the positive even integers which depend on the Bernoulli numbers, the gamma function and the well-known constant π. By taking the appropriate combination of zeta function evaluations, one can arrive at an expression for π which depends only on the zeta function and the gamma function. My first question was whether such a constant exists for the p-adic zeta function. The obvious candidate would be the same expression, with the zeta and gamma functions replaced by their p-adic analogues, but numerical computations of this expression at positive even integers found no structure stronger than continuity, and certainly nothing suggesting a constant.

My next avenue of investigation was to find the zeroes of the p-adic zeta function. As in the case of the classical zeta function, these analytic values are quite significant; in this case, they give arithmetic information about the towers of cyclotomic fields studied in Iwasawa Theory. I used the continuity property of the zeta function given by Kummer's congruences to implement an algorithm that guaranteed to find every zero of a particular p-adic zeta function in the p-adic integers $\mathbf{Z}_p$, and performed a brute-force search for $p < 103$, finding only a few zeroes. These zeroes occurred for irregular primes p, and I later found a proof that these are the only zeroes in $\mathbf{Z}_p$. I manipulated the series expansion to compute the residues at these zeroes, which give information on the behavior of the function near the zero.

To investigate the location of zeroes in a wider subset of $\mathbf{C}_p$, I computed interpolation polynomials that approximate the power series for the p-adic zeta function. For each of these polynomials, I computed the Newton polygon, which allowed me to determine the p-adic valuation ("size") of each zero. Interestingly, the vast majority of the zeroes occurred with valuation 0 (meaning they are on the boundary of the p-adic unit disc), with a few select cases not found in $\mathbf{Z}_p$ having positive non-zero (and often non-integral) valuation, meaning they must lie inside the unit disc of algebraic extensions of $\mathbf{Q}_p$. I did generalize my zero-finding algorithm to work in Galois extensions; but the computational cost increased exponentially with the degree of the extension. If I continue this work in the future, I would like to find more efficient algorithms to approximate these zeroes.

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