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The Lie-Poisson Structure of the Symmetry Reduced Regularised n-Body Problem

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1 Introduction

The n-body problem has the Galilean symmetry group which leads to the classical 9 integrals of linear momentum, centre of mass, and angular momentum. Symplectic reduction of this symmetry leads to a symmetry reduced system with $3n-5$ degrees of freedom, see, e.g. [1]. An alternative approach to reduction that avoids problems with singular reduction uses invariants of the symmetry group action. Singular reduction does occur in the n-body problem because the orbit of the symmetry group drops in dimension for collinear configurations. Using quadratic invariants leads to a Lie-Poisson structure isomorphic fo $\mathfrak{sp}(2n-2)$, as was shown using different basis of invariant in [2] and [3].

One motivation for this approach is the possibility to derive a structure preserving geometric integrator for the symmetry reduced 3-body problem, as derived in [3]. However, numerical integration of many body problems needs to be able to deal with

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binary near-collisions. The classical regularisation by squaring in the complex plane found by Levi-Civita [4] has a beautiful spatial analogue that can be formulated using quaternions [5]. The KS-regularisation has been used by Heggie to simultaneously regularise binary collision in the n-body problem [6]. Recently the symmetry reduction of the regularised 3-body problem has been revisited in [7], extending the classical work of Lemaitre . In the present work we perform the symmetry reduction using quadratic invariants, thus repeating [3] for the regularised problem. Our main result is that the symmetry reduced regularised 3-body problem has a Lie-Poisson structure of the Lie-algebra $\mathfrak{su}(3,3)$.

The paper is organised as follows. In the next section we introduce our notation of quaternions and Heggies regularised Hamiltonian. We then treat the cases $n = 2$ (Kepler), $n = 3$ and $n \geq 4$ in turns. For the Kepler problem we show how to extend the $SO(3)$ group action to a subgroup of $SO(4)$. Treating 3 particles amounts to repeat this construction for 3 difference vectors, and we show that for a suitable chosen group action the space of quadratic invariants is closed and the Hamiltonian can be written in terms of the quadratic invariants. The corresponding Lie-Poisson structure is $\mathfrak{su}(3,3)$. In the final section we briefly comment on how to repeat this construction for an arbitrary number of particles.

2 Simultaneous regularisation of binary collisions

Let the positions of the particles be denoted by $\boldsymbol{q}_i \in \mathbb{R}^3$, and the conjugate momenta by $\boldsymbol{p}_i \in \mathbb{R}^3$, $i = 1, \ldots, n$. The translational symmetry is reduced by forming difference vectors $\boldsymbol{q}_{ij} = \boldsymbol{q}_i - \boldsymbol{q}_j$ and $\boldsymbol{p}_{ij} = \boldsymbol{p}_i - \boldsymbol{p}_j$. We follow [8] in using quaternions for the regularisation. The analogue of Levi-Civita's squaring map can then be written as

$$\boldsymbol{q} = \boldsymbol{Q} * \boldsymbol{Q}^\star, \tag{2.1}$$

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where $\boldsymbol{Q} = Q_0 + \boldsymbol{i}Q_1 + \boldsymbol{j}Q_2 + \boldsymbol{k}Q_3$ and the superscript * flips the sign of the $\boldsymbol{k}$-component, $\boldsymbol{Q}^\star = Q_0 + \boldsymbol{i}Q_1 + \boldsymbol{j}Q_2 - \boldsymbol{k}Q_3$, see [8]. By construction the quaternion $\boldsymbol{Q} * \boldsymbol{Q}^\star$ has vanishing k-component and can thus be identified with the 3-dimensional vector $\boldsymbol{q}$.

The mapping from 4-dimensional momenta $\boldsymbol{P}$ to 3-dimensional momenta $\boldsymbol{p}$ is given by

$$\boldsymbol{p} = \frac{1}{2||\boldsymbol{Q}||^2}\boldsymbol{Q} * \boldsymbol{P}^\star = \frac{1}{2}\boldsymbol{P}^\star * \bar{\boldsymbol{Q}}^{-1} \tag{2.2}$$

where the overbar denotes quaternionic conjugation.

Note that in general the $\boldsymbol{k}$-component of the right hand side is non-zero. One could think of the map to $\boldsymbol{p}$ to be a projection onto the first three components. However, it turns out to be better to impose that the last component vanishes. This condition can be written as

$$\boldsymbol{Q}^T K \boldsymbol{P} = 0 \quad \text{where} \quad K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Here $\boldsymbol{Q}$ and $\boldsymbol{P}$ are interpreted as ordinary 4-dimensional vectors; multiplication of quaternions is denoted by $*$ as above. This is the famous bi-linear relation [5]. Together (2.1) and (2.2) define a projection π from $(\boldsymbol{Q}, \boldsymbol{P}) \in T^*\mathbb{R}^4$ to $(\boldsymbol{q}, \boldsymbol{p}) \in T^*\mathbb{R}^3$. Only when restricting to the subspace defined by the bi-linear relation (2.3) does the map π respect the symplectic structures in the sense that

$$\{f, g\}_3 \circ \pi = \{f \circ \pi, g \circ \pi\}_4 .$$

Here the two Poisson brackets $\{,\}_3$ and $\{,\}_4$ are coming from the two standard symplectic structures $d\boldsymbol{q} \wedge d\boldsymbol{p}$ and $d\boldsymbol{Q} \wedge d\boldsymbol{P}$, respectively.

Using this transformation on the Hamiltonian of the n-body problem written in terms of difference vectors and scaling time gives the regularised Hamiltonian [6]

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$$
\begin{aligned}
H = &\frac{1}{8}\left(\frac{R_{12}R_{31}}{\mu_{23}}\boldsymbol{P}_{23}^T\boldsymbol{P}_{23} + \frac{R_{12}R_{23}}{\mu_{31}}\boldsymbol{P}_{31}^T\boldsymbol{P}_{31} + \frac{R_{23}R_{31}}{\mu_{12}}\boldsymbol{P}_{12}^T\boldsymbol{P}_{12}\right) \qquad (2.4)\\
&-\frac{1}{4}\left(\frac{R_{23}}{m_1}(\boldsymbol{Q}_{31} * \boldsymbol{P}_{31}^\star)^T(\boldsymbol{Q}_{12} * \boldsymbol{P}_{12}^\star) + \frac{R_{31}}{m_2}(\boldsymbol{Q}_{12} * \boldsymbol{P}_{12}^\star)^T(\boldsymbol{Q}_{23} * \boldsymbol{P}_{23}^\star) + \frac{R_{12}}{m_3}(\boldsymbol{Q}_{23}\boldsymbol{P}_{23}^\star)^T * (\boldsymbol{Q}_{31}\boldsymbol{P}_{31}^\star)\right)\\
&-(m_2m_3R_{31}R_{12} + m_3m_1R_{12}R_{23} + m_1m_2R_{23}R_{31} - hR_{23}R_{31}R_{12}).
\end{aligned}
$$

where $R_{ij} = \boldsymbol{Q}_{ij}^T\boldsymbol{Q}_{ij} = \left\|\boldsymbol{q}_{ij}\right\|$ and $\mu_{ij} = \dfrac{m_im_j}{m_i + m_j}$ is the reduced mass of particles i and j.

3 The Kepler Problem $n = 2$

For $n = 2$ there is only a single difference vector $\boldsymbol{q}_{12} = \boldsymbol{q}_1 - \boldsymbol{q}_2$, similarly for $\boldsymbol{p}$. For ease of notation, in this section we are writing $\boldsymbol{q}$ for $\boldsymbol{q}_{12}$, similarly for $\boldsymbol{p}$, and the corresponding quaternions $\boldsymbol{Q}$ and $\boldsymbol{P}$.

The $SO(3)$ symmetry acting on pairs of difference vectors in $\mathbb{R}^3 \times \mathbb{R}^3$ is the diagonal action $\Phi_R : (\boldsymbol{q}, \boldsymbol{p}) \mapsto (R\boldsymbol{q}, R\boldsymbol{p})$ for $R \in SO(3)$. This is a symplectic map whose momentum map is the cross product $\boldsymbol{q} \times \boldsymbol{p}$. Which linear symplectic actions Ψ_S of (subgroups of) $SO(4)$ acting on $\mathbb{R}^4 \times \mathbb{R}^4$ project to Φ_R under π?

Lemma 3.1. *The diagonal action $\Psi_S : (\boldsymbol{Q}, \boldsymbol{P}) \mapsto (S\boldsymbol{Q}, S\boldsymbol{P})$ for $S \in G < SO(4)$ with $G \cong SU(2) \times SO(2)$ projects to the action of Φ_R under π, in other words, the diagram*

$$
\begin{array}{ccc}
T^*\mathbb{R}^3 & \xleftarrow{\ \pi\ } & T^*\mathbb{R}^4 \\
\Phi_R \big\downarrow & & \Psi_S \big\downarrow \\
T^*\mathbb{R}^3 & \xleftarrow{\ \pi\ } & T^*\mathbb{R}^4
\end{array}
$$

commutes.

Proof. Let the rotation $R \in SO(3)$ be given by $R = \exp At$ for some $A \in \text{Skew}(3)$. We assume that G is a topologically closed subgroup of $GL(4)$ so that we can write $S = \exp Bt$ for $B \in \text{Skew}(4)$. The diagram states that $\Phi_R \circ \pi = \pi \circ \Psi_S$. Linearising at the identity, i.e. differentiating with respect to t and setting $t = 0$, and using that Φ_S leaves the norm of quaternions unchanged gives

$$A(\boldsymbol{Q} * \boldsymbol{P}^\star) = (B\boldsymbol{Q}) * \boldsymbol{P}^\star + \boldsymbol{Q} * (B\boldsymbol{P})^\star$$

from the momenta (2.2), and the same equation with $\boldsymbol{P}$ replaced by $\boldsymbol{Q}$ from the positions (2.1). For arbitrary given $A = \hat{L}$ where $L = (L_x, L_y, L_z)^t$ and the hat-map from $\mathbb{R}^3$ to Skew(3), the general solution can be written as $B = \frac{1}{2}\operatorname{Isoc}(\hat{L}) + \tau K$, where $\operatorname{Isoc}(\hat{L}) = \begin{pmatrix} \hat{L} & -L \\ L^t & 0 \end{pmatrix}$. The subgroup $G < SO(4)$ contains the subgroup of isoclinic rotations $\exp(\operatorname{Isoc}(A)) = \cos\omega\, I_4 + \omega^{-1}\sin\omega \operatorname{Isoc}(A)$ where $\omega^2 = \frac{1}{2}\operatorname{Tr} AA^t$. They form a subgroup since the corresponding generators $\operatorname{Isoc}(A)$ form an algebra with $[\operatorname{Isoc}(\hat{x}), \operatorname{Isoc}(\hat{y})] = 2\operatorname{Isoc}([\hat{x}, \hat{y}]) = 2\operatorname{Isoc}(\widehat{x \times y})$. The corresponding group of left-isoclinic rotation matrices $\exp(\operatorname{Isoc}(\hat{x}))$ has a composition law given by left-multiplication of unit quaternions with imaginary part proportional to x, and hence is isomorphic to $S^3 \cong SU(2)$. The whole group G is obtained by multiplying the general left-isoclinic multiplication $\exp(\operatorname{Isoc}(A))$ with the special right-isoclinic multiplication $\exp(K\tau)$. These two commute, since $\operatorname{Isoc}(A)$ and K commute. The group $\exp(K\tau)$ is isomorphic to $SO(2)$.

□

Lemma 3.2. *The group action Ψ_S has momentum map $L = \Im(\boldsymbol{Q} * \bar{\boldsymbol{P}})$ and $L_\tau = \boldsymbol{Q}^T K \boldsymbol{P}$ which are mapped into the Lie algebra $\mathfrak{g}$ by $\frac{1}{2}\operatorname{Isoc}(\hat{L}) + L_\tau$. When in addition, the bilinear relation is imposed then $\pi \circ L$ becomes the ordinary momentum $\boldsymbol{q} \times \boldsymbol{p}$.*

Here $\Im$ applied to a quaternion takes its i, j, k components.

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Proof. The natural definition of the components of the angular momentum is $L_\alpha = q^T Isoc(\hat{\alpha})p$ where $\alpha \in \{x, y, z, \}$. These components indeed form the imaginary components of $\boldsymbol{Q} * \bar{\boldsymbol{P}}$. For example, $q^T Isoc(\hat{y})p = Q_1P_3 - P_1Q_3 + Q_2P_4 - P_2Q_4$ is the j-component of the quaternion $\boldsymbol{Q} * \bar{\boldsymbol{P}}$. The second statement is shown through direct computation. See [9] for more details.

□

Lemma 3.3. *The basic polynomial invariants of the group action Ψ_S of G are*

$$X_1 = \boldsymbol{Q}^T\boldsymbol{Q}, \quad X_2 = \boldsymbol{P}^T\boldsymbol{P}, \quad X_3 = \boldsymbol{Q}^T\boldsymbol{P}, \quad X_4 = \boldsymbol{P}^T K\boldsymbol{Q}\,.$$

The Poisson bracket of these invariants is closed.

Proof. Firstly, $SO(4)$, as the group of rotations preserves the inner product on $\mathbb{R}^4$. Thus, G as subgroup of SO(4) must also preserve the inner product. Hence, the first three are clearly invariants. Furthermore, denoting $exp(Isoc(\alpha))$ as $R(\alpha)$ where $\alpha \in \{\hat{x}, \hat{y}, \hat{z}\}$, it is easily shown that $R(\alpha)^T K R(\alpha) = K\ \forall\alpha$. Also, $exp(K)^T K exp(K) = K$. Thus,Ψ_s preserves quadratic forms over K and hence the fourth quantity is also an invariant. Furthermore, it can be shown by direct computation that the only matrices, M, that satisfy $\{x^T Mx, L_\alpha\} = 0$ where $x \in \{\mathbf{Q}, \mathbf{P}\}$, for all $\alpha \in \{x, y, z, \tau\}$ are linear combinations of K and the identity matrix. Therefore, the above set is a basis for our vector space of quadratic invariants. The Poisson algebra has the following structure:

$$\{X_i, X_4\} = 0 \qquad \forall i \in \{1, 2, 3, 4\} \tag{3.1}$$

$$\{X_i, X_3\} = 2X_i \qquad \forall i \in \{1, 2\} \tag{3.2}$$

$$\{X_1, X_2\} = 4X_3 \tag{3.3}$$

It follows that vector space generated by these four invariants are closed under the Poisson bracket.

□

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Under the basis (X_1, X_2, X_3, X_4), the Poisson structure matrix is

$$\begin{pmatrix} 0 & 4X_3 & 2X_1 & 0 \\ -4X_3 & 0 & -2X_2 & 0 \\ -2X_1 & 2X_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, Lie algebra of the quadratic invariants is actually isomorphic to $\mathfrak{u}(1,1)$. Refer to [9] for details.

4 The 3-body problem $n = 3$

The action Ψ_S on pairs $(\boldsymbol{q}, \boldsymbol{p})$ extends to an action (denoted by the same letter) on triples of pairs $(\boldsymbol{q}_{ij}, \boldsymbol{p}_{ij})$. Since the action is diagonal, to get the corresponding angular momenta the individual momenta are simply added together, $\mathcal{L}_a = \sum L_a^i$ for $a \in \{x, y, z, \tau\}$.

Choosing the correct symmetry group is crucial in order to obtain a good set of quadratic invariants. When enlarging the group by taking each L_τ^i separately instead of their sum as generators the set of quadratic invariants has only 9 elements. However, Heggies' Hamiltonian cannot be written in terms of these 9 invariants. The present choice of Ψ_S gives the smallest set of closed quadratic invariants in terms of which the Hamiltonian can be expressed.

Lemma 4.1. *The set of quadratic forms Q invariant under Ψ_S is of the form $Q = (X, MX)$ with*

$$M = [W]_{sym} \otimes I_4 + [W]_{skew} \otimes K$$

where W is an arbitrary 6×6 matrix, $X = (\boldsymbol{Q}_1^T, \boldsymbol{Q}_2^T, \boldsymbol{Q}_3^T, \boldsymbol{P}_1^T, \boldsymbol{P}_2^T, \boldsymbol{P}_3^T)$ and $\otimes$ denotes the Kronecker product. The space of quadratic invariants is closed under the Poisson bracket and hence form a Lie algebra $\mathfrak{g}$.

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Proof. As in Lemma 3.3, the inner products and the quadratic forms over K are the invariants under Ψ_S. Representing these as quadratic forms over the 24 dimensional phase space, it is clear that the matrices of the inner products have the form $E \otimes I_4$ where E is an element of the standard basis of $sym(6)$. Similarly, the quadratic forms over K can be represented as $F \otimes K$ where F is an element of the standard basis of $skew(6)$. As the sum of invariants is invariant, the matrix for any quadratic invariant can be written as $S \otimes I_4 + A \otimes K$ where $S \in sym(6), A \in skew(6)$. Thus, the set of quadratic invariants is of the form $Q = (X, MX)$ where

$$M = [W]_{sym} \otimes I_4 + [W]_{skew} \otimes K$$

where W is an arbitrary 6×6 matrix and so the space of quadratic invariants is isomorphic to $Mat(6 \times 6, \mathbb{R})$ as vector spaces. The Poisson bracket of two inner products and that of two quadratic forms over K are linear combinations of inner products while the Poisson bracket of an inner product and a quadratic form over K is a linear combination of quadratic forms over K. Thus, the Poisson bracket is closed. For example, defining $\alpha_{i,j} = Q_i^T Q_j, \quad \beta_{i,j} = P_i^T P_j, \quad \gamma_{i,j} = Q_i^T P_j, \quad a_{i,j} = Q_i^T K Q_j, \quad b_{i,j} = P_i^T K P_j, \quad c_{i,j} = Q_i^T K P_j$, we have that $\{\alpha_{1,1}, \beta_{1,1}\} = 4\gamma_{1,1}$, while, $\{\alpha_{1,1}, c_{3,1}\} = -2a_{1,3}$. Furthermore, this implies that the 21-dimensional subspace generated by all possible combinations of inner products is closed under the Poisson bracket and hence form a subalgebra.

□

Let $f_{ij} = 4(\gamma_{i,j}\gamma_{j,i} - \gamma_{i,i}\gamma_{j,j} + \beta_{i,j}\alpha_{i,j} - c_{i,j}c_{j,i} + b_{i,j}a_{i,j})$. The Hamiltonian in terms of

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the invariant quadratic forms reads

$$H = \frac{1}{8}\left(\frac{\alpha_{2,2}\alpha_{3,3}}{\mu_{23}}\beta_{1,1} + \frac{\alpha_{3,3}\alpha_{1,1}}{\mu_{13}}\beta_{2,2} + \frac{\alpha_{1,1}\alpha_{2,2}}{\mu_{12}}\beta_{3,3}\right)$$
$$- \frac{1}{16}\left(\frac{\alpha_{1,1}}{m_1}f_{23} + \frac{\alpha_{2,2}}{m_2}f_{13} + \frac{\alpha_{3,3}}{m_3}f_{12}\right)$$
$$- m_2 m_3 \alpha_{2,2}\alpha_{3,3} - m_1 m_3 \alpha_{1,1}\alpha_{3,3} - m_1 m_2 \alpha_{1,1}\alpha_{2,2} - h\alpha_{1,1}\alpha_{2,2}\alpha_{3,3}\,.$$

Now in order to work out the isomorphism type of the Lie algebra of quadratic invariants, we need to induce Lie bracket on $Mat(6 \times 6, \mathbb{R})$. In general we have the following:

Lemma 4.2. *If $\varphi : X \to Y$ is a vector space isomorphism between the Lie algebra $(X, [\cdot,\cdot]_X)$ and the vector space Y (both over the same field) then $[y_1,y_2]_Y = \varphi([\varphi^{-1}(y_1), \varphi^{-1}(y_2)]_X)$ defines a Lie bracket on Y and hence $(X, [\cdot,\cdot]_X)$ and $(Y, [\cdot,\cdot]_Y)$ are isomorphic as Lie algebras under φ.*

Proof. As φ is an isomorphism, both φ and φ^{-1} are linear and bijective so we have:

1. Bilinearity: $[\alpha y_1 + \beta y_2 \gamma, y_3 + \delta y_4]_Y = \varphi([\varphi^{-1}(\alpha y_1 + \beta y_2), \varphi^{-1}(\gamma y_3 + \delta y_4)]_X = \alpha\gamma[y_1, y_3]_Y + \alpha\delta[y_1, y_4]_Y + \beta\gamma[y_2, y_3]_Y + \beta\delta[y_2 + y_4]_Y$ by linearity of φ , φ^{-1} and bilinearity of $[\cdot,\cdot]_X$.

2. Alternating: $[y, y]_Y = \varphi([\varphi^{-1}(y), \varphi^{-1}(y)]_X) = \varphi(0) = 0$ as $[\cdot,\cdot]_X$ is alternating and $0 \in \ker(\varphi)$.

3. Jacobi Identity: $[y_1, [y_2, y_3]] + [y_3, [y_1, y_2]] + [y_2, [y_3, y_1]] = \varphi([[\varphi^{-1}(y_1), [[\varphi^{-1}(y_2), [\varphi^{-1}(y_3)]] + [[\varphi^{-1}(y_3), [[\varphi^{-1}(y_1), [\varphi^{-1}(y_2)]] + [[\varphi^{-1}(y_2), [[\varphi^{-1}(y_3), [\varphi^{-1}(y_1)]]) = \varphi(0) = 0$ by Jacobi Identity of $[\cdot,\cdot]_X$ and linearity of φ .

□

Since we have quadratic invariants, the new Poisson bracket induces an algebra on the form of the quadratic invariants. There is a bijection between the space of quadratic

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functions over the our phase space ($(T^*\mathbb{R}^4)^3 \cong \mathbb{R}^{24}$) and the vector space of 24×24 symmetric matrices. Therefore we get an isomorphism from a subspace U of $Sym(24,\mathbb{R})$ and the vector space generated by the 36 quadratic invariants (denote this Z) given by:

$$f : U \to Z; \qquad M \mapsto \frac{1}{2}\langle \mathbf{X}, M\mathbf{X}\rangle$$

where $X = (\mathbf{Q}_1, ..., \mathbf{Q}_4, \mathbf{P}_1, ..., \mathbf{P}_4)$ is the vector of the 24 variables (order in a sensible manner...). We can turn U into a Lie Algebra by defining the Lie bracket:

$$[\cdot,\cdot]_f : U \times U \to U; \qquad [M,N]_f = MJN + (MJN)^T = MJN - NJM = 2[MJN]_{sym}.$$

It is well known that this algebra is isomorphic to $\mathfrak{sp}(m)$ [3], where in our case $m = 24$.

From Lemma 4.1 and Lemma 4.2 the isomorphism $m : U \to Mat(6\times 6,\mathbb{R})$ given by $m(\tilde{A}\otimes I_4 + \check{A}\otimes K) = \tilde{A} + \check{A} := A$ induces a Lie Bracket $[\cdot,\cdot]_m$ on $Mat(6\times 6,\mathbb{R})$. We have

$$\begin{aligned}
[m^{-1}(A), m^{-1}(B)]_f &= [m^{-1}(A)J_{24}\, m(B)]_{sym} = [m^{-1}(A)(J_6\otimes I_4)m^{-1}(B)]_{sym}\\
&= [(\tilde{A}\otimes I_4 + \check{A}\otimes K)(J\otimes I_4)(\tilde{B}\otimes I_4 + \check{B}\otimes K)]_{sym}\\
&= [(\tilde{A}J\tilde{B} - \check{A}J\check{B})\otimes I_4 + (\check{A}J\tilde{B} + \tilde{A}J\check{B})\otimes K]_{sym}\\
&= (\tilde{A}J\tilde{B} - \tilde{B}J\tilde{A} - \check{A}J\check{B} + \check{B}J\check{A})\otimes I_4 + (\check{A}J\tilde{B} - \tilde{B}J\check{A} + \tilde{A}J\check{B} + \check{B}J\tilde{A})\otimes K\\
&= -J([J\tilde{A}, J\tilde{B}] - [J\check{A}, J\check{B}])\otimes I_4 - J([J\tilde{A}, J\check{B}] + [J\check{A}, J\tilde{B}])\otimes K
\end{aligned}$$

where $\tilde{(\cdot)} = [\cdot]_{sym}$ and $\check{(\cdot)} = [\cdot]_{skew}$. Hence $[A,B]_m = -J([J\tilde{A}, J\tilde{B}] - [J\check{A}, J\check{B}] + [J\tilde{A}, J\check{B}] + [J\check{A}, J\tilde{B}])$ is the induced bracket on $Mat(6\times 6,\mathbb{R})$. Now we are ready for the core theorem of this paper:

Theorem 4.1. *The symmetry reduced regularised 3-body problem has a Lie-Poisson structure with algebra $\mathfrak{u}(3,3)$ and a corresponding Hilbert basis of 36 quadratic functions invariant under Ψ_S.*

Proof. For $A \in Mat(6 \times 6, \mathbb{R})$ then the matrix $a = J(\tilde{A} + i\check{A})$ is in $\mathfrak{u}(3,3)$, as the hermitian matrix $H = -iJ$ has eigenvalues ± 1 each with multiplicity 3 and we have $(HM)^\dagger + HM = 0$. Therefore we have a vector space isomorphism $h : Mat(6 \times 6, \mathbb{R}) \to \mathfrak{u}(3,3)$ with $h(A) = -2iJ(\tilde{A} + i\check{A}) = a$. We now compute the induced bracket on the vector space of $\mathfrak{u}(3,3)$ under h, $[\cdot,\cdot]_h$ and show that this coincides with the standard commutator (which is the Lie bracket on $\mathfrak{u}(3,3)$). We have:

$$[h^{-1}(a), h^{-1}(b)]_m = -J([J\tilde{A}, J\tilde{B}] - [J\check{A}, J\check{B}] + [J\tilde{A}, J\check{B}] + [J\check{A}, J\tilde{B}])$$

with $[[h^{-1}(a), h^{-1}(b)]_m]_{sym} = -J([J\tilde{A}, J\tilde{B}] - [J\check{A}, J\check{B}])$ and $[[h^{-1}(a), h^{-1}(b)]_m]_{skew} = -J([J\tilde{A}, J\check{B}] + [J\check{A}, J\tilde{B}])$. Hence

$$\begin{aligned}
[a,b]_h &= h\left([h^{-1}(a), h^{-1}(b)]_m\right) \\
&= J\left(-J\left([J\tilde{A}, J\tilde{B}] - [J\check{A}, J\check{B}]\right) - iJ\left([J\tilde{A}, J\check{B}] + [J\check{A}, J\tilde{B}]\right)\right) \\
&= [J\tilde{A}, J\tilde{B}] + [Ji\check{A}, Ji\check{B}] + [J\tilde{A}, Ji\check{B}] + [Ji\check{A}, J\tilde{B}] \\
&= [J(\tilde{A} + i\check{A}), J(\tilde{B} + i\check{B})] \\
&= [a,b]\,.
\end{aligned}$$

This proves that space of quadratic invariants and $\mathfrak{u}(3,3)$ are isomorphic as Lie algebra.

□

Reduction by the centre of the algebra which is generated by $\mathcal{L}_\tau$ gives $\mathfrak{su}(3,3)$.

Lemma 4.3. *The Poisson structure has 6 Casimirs of degree 1 through 6. The linear Casimir is the sum of the bilinear integrals $\mathcal{L}_\tau$, the quadratic Casimir is the sum of the three angular momenta squared $\mathcal{L}_x^2 + \mathcal{L}_x^2 + \mathcal{L}_z^2$.*

Proof. The Poisson bracket of the Lie algebra, in this matrix representation, can be written as

$$\{f,g\}(M) = \left\langle M, \left[\frac{df}{dM}, \frac{dg}{dM}\right]\right\rangle$$

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where, the inner product is given by $< M, N >= \text{Tr}(M^\dagger M)$ and $\frac{df}{dM}$ refers to the element in $\mathfrak{g}$ that satisfies

$$\lim_{e->0}[f(M+edM)-f(M)] =< dM, \frac{df}{dM} >$$

See [10] for more details. It can be shown that the co-effiecients of the characteristic polynomial of $J(\tilde{A}+i\check{A})$ are in fact the Casimirs under this Poisson bracket. The co-efficient of the fifth order term is just the sum of the bilinear integrals, $\mathcal{L}_\tau$. The co-efficient of the quartic term is of the form:

$$\mathcal{L}_x^2+\mathcal{L}_y^2+\mathcal{L}_z^2+f(\mathcal{L}_\tau)$$

where $f(\mathcal{L}_\tau)$ is a quadratic function of the bilinear integrals. Under the reduction by the centre, this Casimir simply becomes $\mathcal{L}_x^2+\mathcal{L}_y^2+\mathcal{L}_z^2$.

□

The fact that the three difference vectors $\boldsymbol{q}_{ij}$ add to zero induces another three quadratic integrals T_1, T_2, T_3. The flow of these integrals is non-compact, and we were not able to use it for symmetry reduction. The three momenta and the integrals T_i form the Algebra $\mathfrak{se}(3)$.

5 The n-body problem

Theorem 5.1. *The symmetry reduced regularised n-body problem has a Lie-Poisson structure with algebra $\mathfrak{u}(m,m)$ where $m = n(n-1)/2$.*

Proof. As shown in Lemma 4.1, the nature of the invariants under Ψ_S are independent of the number of particles. They are realised as in the aforementioned lemma in phase space by the use of symmetric and antisymmetric matrices of size $2m \times 2m$ where m

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denotes the number of difference vectors in the system. This establishes the vector space isomorphism to the space of $2m \times 2m$ matrices. Furthermore, by Theorem 4.1, it is apparent that the Lie algebra of invariants is isomorphic to $\mathfrak{u}(m, m)$. As m is equal to $\binom{n}{2} = n(n-1)/2$, the algebra of invariants for the symmetry reduced regularised n-body problem has a Lie-Poisson structure with algebra $\mathfrak{u}(n(n-1)/2, n(n-1)/2)$.

□

6 Conclusion

In this paper, we have shown that the quadratic invariants of the regularised n-body problem are either inner products or quadratic forms over the antisymmetric matrix K. These invariants form a Lie-Poisson algebra that is isomorphic to the lie algebra $\mathfrak{su}(m, m)$ where $m = n(n-1)/2$ which is the algebra corresponding to the group that preserves hermitian forms of signature (m, m). The dimension of this Lie Algebra is of order n^4. Thus the use of such an algebra to obtain numerical solutions is improbable for large values of n. Despite this, the isomorphism to $\mathfrak{su}(m, m)$ yields a large amount of information about the rich structure of these invariants and provides insight into the n-body problem.

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