de Rham's Theorem via Čech Cohomology

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February 2014

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1 Introduction

The Euler Characteristic of a shape is what is known as a topological invariant. That is, under continuous deformation of our shape, we will always arrive at a shape with the same Euler Characteristic. We can therefore deduce that two shapes that do not share the same Euler Characteristic can not be deformed to one another continuously. The idea of a topological invariant is useful in classifying spaces and we look for structures that preserve the most information about spaces that are equivalent via continuous deformation.

The de Rham cohomology is a structure, specifically a vector space, on a space based upon your ability to solve differential equations on it that is also a topological invariant. In 1931 Georges de Rham proved a result that relates the de Rham cohomology to a much more abstract cohomology. The result as stated in 1931 is very different from the result as interpreted today since the notion of cohomology had not been formed yet. The purpose of this paper will be to prove a specific case of the theorem as interpreted today, namely the isomorphism between what is known as the Čech cohomology and the de Rham cohomology. In order to do this, we first introduce the notion of cohomology, then prove some fundamentals of the de Rham cohomology, after which we form the necessary relations between the de Rham cohomology and Čech cohomology to complete the proof.

2 The de Rham Complex

2.1 Multivariable Calculus

When doing calculus in $\mathbb{R}^3$ we often come across the operators $Grad$, $Curl$ and Div. Given $U \subset \mathbb{R}^3$ and smooth functions $f, f_1, f_2, f_3 : U \to \mathbb{R}$ we define:

$$\begin{gathered} Grad(f) = \nabla f = \langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z} \rangle, \\ Curl(\langle f_1, f_2, f_3 \rangle) = \nabla \times \langle f_1, f_2, f_3 \rangle = \langle \tfrac{\partial f_3}{\partial y} - \tfrac{\partial f_2}{\partial z}, \tfrac{\partial f_1}{\partial z} - \tfrac{\partial f_3}{\partial x}, \tfrac{\partial f_2}{\partial x} - \tfrac{\partial f_1}{\partial y} \rangle, \\ \text{and } Div(\langle f_1, f_2, f_3 \rangle) = \nabla \cdot \langle f_1, f_2, f_3 \rangle = \tfrac{\partial f_1}{\partial x} + \tfrac{\partial f_2}{\partial y} + \tfrac{\partial f_3}{\partial z}. \end{gathered}$$

A helpful way of visualising these operators is the following diagram:

$$\left\{ \begin{matrix} \text{functions} \\ \text{from } U \text{ to } \mathbb{R} \end{matrix} \right\} \xrightarrow{\nabla} \left\{ \begin{matrix} \text{vector fields} \\ \text{on } U \text{ over } \mathbb{R}^3 \end{matrix} \right\} \xrightarrow{\nabla \times} \left\{ \begin{matrix} \text{vector fields} \\ \text{on } U \text{ over } \mathbb{R}^3 \end{matrix} \right\} \xrightarrow{\nabla \cdot} \left\{ \begin{matrix} \text{functions} \\ \text{from } U \text{ to } \mathbb{R} \end{matrix} \right\}$$

where we observe that the composition of two consecutive operations will give us 0, i.e. $Curl \circ Grad = 0$ and $Div \circ Curl = 0$. This is equivalent to saying $\text{im}(Grad) \subset \ker(Curl)$

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and $\mathrm{im}(Curl) \subset \ker(Div)$. The natural question to ask is under what conditions these two inclusions hold as equalities. Before answering this question we will first try to generalise our notions of *Grad*, *Curl* and *Div* to higher dimensions.

2.2 Differential Forms

Let $\mathbb{R}^n$ have the standard basis $\mathfrak{B} = \{x_1, x_2, ..., x_n\}$. We firstly denote the smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$ by the *0-forms*, $\mathbf{\Omega^0}(\mathbb{R}^n)$. The notion of *Grad* can be easily extended to the functions in $\mathbf{\Omega^0}(\mathbb{R}^n)$ by:

$$Grad(f) = \langle \tfrac{\partial f}{\partial x_1}, \tfrac{\partial f}{\partial x_2}, ..., \tfrac{\partial f}{\partial x_n} \rangle$$

For reasons that will become clear, we will instead denote this analog of *Grad* by d and write it as:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \ .$$

In general a *1-form* is anything of the form:

$$\omega = \sum_{i=1}^{n} f_i \, dx_i$$

where $f_i \in \mathbf{\Omega^0}(\mathbb{R}^n)$ for $i = 1, 2, ..., n$. The set of all 1-forms defined on $\mathbb{R}^n$ is denoted $\mathbf{\Omega^1}(\mathbb{R}^n)$. We can integrate a 1-form over a curve in n-dimensional space quite naturally too.

Example 2.1: Let $\omega = \sum_{i=1}^{n} f_i \, dx_i \in \mathbf{\Omega^1}(\mathbb{R}^n)$ and C be a curve in $\mathbb{R}^n$ parametrised by $\gamma : [0, 1] \to \mathbb{R}^n$, $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$. Then

$$\begin{aligned}
\int_C \omega &= \int_C \sum_{i=1}^{n} f_i \, dx_i \\
&= \int_0^1 \sum_{i=1}^{n} f_i \circ \gamma \, d\gamma_i \\
&= \int_0^1 \sum_{i=1}^{n} f_i(\gamma(t))\gamma_i'(t) \, dt
\end{aligned}$$

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Example 2.2: Let $\omega = 2x\,dx + 3\,dy \in \mathbf{\Omega^1}(\mathbb{R}^2)$ and C be a curve in $\mathbb{R}^2$ parametrised by $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = (t^2, 2t)$. Then

$$\begin{aligned}
\int_C \omega &= \int_C 2x\,dx + 3\,dy \\
&= \int_0^1 2t^2\,d(t^2) + 3\,d(2t) \\
&= \int_0^1 4t^3\,dt + 6\,dt \\
&= \int_0^1 4t^3 + 6\,dt \\
&= \left[t^4 + 6t\right]_0^1 \\
&= 7
\end{aligned}$$

Whilst *Grad* is easily extended to a higher dimension, the notion of *Curl* on a vector field $\langle f, g, h\rangle$ is often motivated by taking the psuedo-determinant of the 3x3 matrix:

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{pmatrix}$$

which can't be extended to higher dimensions in any obvious manner. To counter this, we introduce the notion of a *2-form*. A 2-form on $\mathbb{R}^n$ is an object of the form

$$\omega = \sum_{i,j=1}^{n} f_{ij}\,dx_i \wedge dx_j \in \mathbf{\Omega^2}(\mathbb{R}^n)$$

where the *wedge product*, $\wedge$, is a similar operation to the cross-product on $\mathbb{R}^3$ in the sense that:

$$\begin{aligned} dx \wedge dx &= 0 \\ dx \wedge dy &= -dy \wedge dx \end{aligned}$$

In this construction, we treat the functions as 'linear coefficients' in that:

$$(f\,dx) \wedge dy = f(dx \wedge dy) = dx \wedge f(dy)$$

We can now extend d to act on a 1-form, ω, by defining:

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$$d\omega = d\left(\sum_{i=1}^{n} f_i\,dx_i\right) = \sum_{i=1}^{n} df_i \wedge dx_i$$

It is not immediately obvious that d is an analog of $Curl$, but we note that given $\omega = f_1\,dx + f_2\,dy + f_3\,dz \in \mathbf{\Omega^1}(\mathbb{R}^3)$ where $f_1, f_2, f_3 \in \mathbf{\Omega^0}(\mathbb{R}^3)$

$$\begin{aligned}
d\omega =& d(f_1\,dx + f_2\,dy + f_3\,dz) \\
=& df_1 \wedge dx + df_2 \wedge dy + df_3 \wedge dz \\
=& \left(\frac{\partial f_1}{\partial x}\,dx \wedge dx + \frac{\partial f_1}{\partial y}\,dy \wedge dx + \frac{\partial f_1}{\partial z}\,dz \wedge dx\right) \\
&+ \left(\frac{\partial f_2}{\partial x}\,dx \wedge dy + \frac{\partial f_2}{\partial y}\,dy \wedge dy + \frac{\partial f_2}{\partial z}\,dz \wedge dy\right) \\
&+ \left(\frac{\partial f_3}{\partial x}\,dx \wedge dz + \frac{\partial f_3}{\partial y}\,dy \wedge dz + \frac{\partial f_3}{\partial z}\,dz \wedge dz\right) \\
=& \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy
\end{aligned}$$

which, as you may have noticed, looks strikingly similar to

$$Curl(\langle f_1, f_2, f_3\rangle) = \left\langle \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right\rangle$$

The elements of $\mathbf{\Omega^2}(\mathbb{R}^n)$ form a module over the smooth functions, $C^\infty(\mathbb{R}^n, \mathbb{R})$, since we can add forms together and multiply them by smooth functions. We can therefore form a basis for $\mathbf{\Omega^2}(\mathbb{R}^n)$, namely:

$$\{dx_i \wedge dx_j : x_i, x_j \in \mathfrak{B}, i < j\}.$$

In general we have *q-forms* which are of the form

$$\omega = \sum_{i_1,\ldots,i_q} f_{i_1\cdots i_q}\,dx_{i_1} \wedge \cdots \wedge dx_{i_q} \in \mathbf{\Omega^q}(\mathbb{R}^n).$$

Once again, these form a basis over $C^\infty(\mathbb{R}^n, \mathbb{R})$ where the basis of $\mathbf{\Omega^q}(\mathbb{R}^n)$ can be written

$$\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q} : x_{i_1}, ..., x_{i_q} \in \mathfrak{B}, i_1 < \cdots < i_q\}.$$

Remark: Because of the rule $dx_i \wedge dx_j = -dx_j \wedge dx_i$, differential forms that have the same indices are not linearly independent. To correct this, so as to form a basis, we order the indices.

From this basis on the q-forms it is evident that if $q > n$ then $\mathbf{\Omega^q}(\mathbb{R}^n) = 0$. We can think of $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_q$ as an infinitesimal, signed notion of n-dimensional volume over which we can integrate given a parametrisation. The wedge product be used on arbitrary forms rather than just the dx elements, so we define the notion of the wedge product on two differential forms.

Definition 2.1: *Let* $\omega = \sum f\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \in \mathbf{\Omega^p}(\mathbb{R}^n)$ *and* $\psi = \sum g\, dx_{j_1} \wedge \cdots \wedge dx_{j_q} \in \mathbf{\Omega^q}(\mathbb{R}^n)$. *We define* $\wedge : \mathbf{\Omega^p}(\mathbb{R}^n) \times \mathbf{\Omega^q}(\mathbb{R}^n) \to \mathbf{\Omega^{p+q}}(\mathbb{R}^n)$ *by*

$$\omega \wedge \psi = \sum f \cdot g\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}$$

The natural extension of $d : \mathbf{\Omega^q}(\mathbb{R}^n) \to \mathbf{\Omega^{q+1}}(\mathbb{R}^n)$ arises by defining

$$d\omega = \sum_{i_1,\ldots,i_q} df_{i_1 \cdots i_q} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \in \mathbf{\Omega^{q+1}}(\mathbb{R}^n)$$

which is compatible with our previous understanding of d. In order to see if our extension of d is also an analog of Div, suppose we have a 2-form, $\omega = f_1\, dy \wedge dz - f_2\, dx \wedge dz + f_3\, dx \wedge dy$. Then

$$\begin{aligned} d\omega &= df_1 \wedge dy \wedge dz - df_2 \wedge dx \wedge dz + df_3 \wedge dx \wedge dy \\ &= \frac{\partial f_1}{\partial x}\, dx \wedge dy \wedge dz - \frac{\partial f_2}{\partial y}\, dy \wedge dx \wedge dz + \frac{\partial f_3}{\partial z}\, dz \wedge dx \wedge dy \\ &= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx \wedge dy \wedge dz \end{aligned}$$

which is as we expect.

Theorem 2.1: $d^2 = 0$.

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Proof. Since d is linear it will be sufficient to show this for the case $\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_q}$

$$\begin{aligned} d^2\omega =& d(df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}) \\ =& d\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \right) \\ =& \sum_{j=1}^{n} d\left(\frac{\partial f}{\partial x_j} \right) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ =& \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}\, dx_k \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ =& \sum_{j>k} \frac{\partial^2 f}{\partial x_j \partial x_k}\, dx_k \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ & - \sum_{j<k} \frac{\partial^2 f}{\partial x_j \partial x_k}\, dx_j \wedge dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \end{aligned}$$

In this last step we break the summand into the component where $j > k$ and where $j < k$ and then swap the $dx_k \wedge dx_j$ of latter so that they are in order. We then note that with a change of variables in the second sum, namely swapping j and k, we can further simplify to:

$$d^2\omega = \sum_{j>k} \left(\tfrac{\partial^2 f}{\partial x_j \partial x_k} - \tfrac{\partial^2 f}{\partial x_k \partial x_j} \right) dx_k \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

And thus, by the commutativity of partial derivatives on smooth functions, we have that $d^2\omega = 0$ for every differential form. Hence, $d^2 = 0$. ☐

2.3 The de Rham Complex

The collection of all differentiable forms on $\mathbb{R}^n$ is denoted *the de Rham Complex on* $\mathbb{R}^n$

$$\boldsymbol{\Omega}^*(\mathbb{R}^n) = \bigoplus_q \boldsymbol{\Omega}^q(\mathbb{R}^n).$$

The domain $\mathbb{R}^n$ is superficial and could easily be replaced by a subset $U \subset \mathbb{R}^n$, however. A form defined on $U \subset \mathbb{R}^n$ has a larger set of smooth functions to call on. For example, if we let $V = \mathbb{R}^2 \backslash \{(0,0)\}$, the differential form

$$\omega = \tfrac{y}{x^2+y^2}\, dx + \tfrac{x}{x^2+y^2}\, dy$$

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cannot be defined on $\mathbb{R}^2$, but can be defined on V. We denote the set of all differentiable *q-forms* on $U \subset \mathbb{R}^n$ by $\mathbf{\Omega^q}(U)$ and similarly the collection of all differentiable forms on U is denoted *the de Rham Complex on U*

$$\mathbf{\Omega^*}(U) = \bigoplus_q \mathbf{\Omega^q}(U).$$

To ensure that d makes sense, we will want to be careful in our choice of U. We will require that our space looks locally similar enough to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ so that we can perform calculus on it. In the language of topology, we will require a *smooth manifold*: a manifold with a *differentiable structure*.

Definition 2.2: *A* differentiable structure *on an n*-dimensional manifold *M is given by an* atlas, *i.e., an open cover* $\{U_\alpha\}_{\alpha\in A}$ *of M in which each open set* U_α *is homeomorphic to* $\mathbb{R}^n$ *via a homeomorphism* $\phi_\alpha : U_\alpha \to \mathbb{R}^n$*, and on the overlaps* $U_\alpha \cap U_\beta$ *the transition functions*

$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are diffeomorphisms of open subsets of $\mathbb{R}^n$*. All manifolds will be assumed to be Hausdorff and to have countable basis.*

This definition is taken from Bott & Tu [1, p.20], but for those interested in a deeper understanding of manifolds we refer the reader to Conlon [3]. The theory of differential forms and the de Rham complex on $\mathbb{R}^n$ naturally extends to the setting of manifolds but for more details it is strongly recommended that you read Bott & Tu [1] for further details.

Definition 2.3: *Let M be an n-dimensional manifold and* $\omega \in \mathbf{\Omega^q}(M)$*. We say that* ω *is* closed *if* $d\omega = 0$ *and* exact *if there exists* $\phi \in \mathbf{\Omega^{q-1}}(M)$ *such that* $\omega = d\phi$

A closed form is hence an element of the kernel of d and an exact form is an element of the image of d. A corollary of theorem 1.1 is that every exact form is closed. This is a property that is familiar to a class of objects called *differential complexes.*

Definition 2.4: *A direct sum of vector spaces* $C = \bigoplus_{q\in\mathbb{Z}} C^q$ *is called a* differential complex *if there are linear maps* $\partial_i : C^i \to C^{i+1}$ *such that* $\partial_{i+1} \circ \partial_i = 0$*. We call* ∂_i *a differential operator of the complex C.*

$$\cdots \longrightarrow C^{q-1} \xrightarrow{\partial_{q-1}} C^q \xrightarrow{\partial_q} C^{q+1} \longrightarrow \cdots$$

Definition 2.5: *Let c be an element of a differential complex C with differential operators $\partial_i : C^i \to C^{i+1}$. We say c is a p-*cocycle *if $c \in \ker(\partial_p)$ and a p-*coboundary *if $c \in \operatorname{im}(\partial_{p-1})$.*

Corollary 2.2: *$\mathbf{\Omega^*}(M)$ is a differential complex with differential operator d for a given manifold M.*

Remark: We let $\mathbf{\Omega^q}(M) = 0$ for all $q < 0$.

2.4 Cohomology

In the more general setting of a differential complex we find that $\operatorname{im}(\partial_{q-1}) \subset \ker(\partial_q)$. The natural question to ask is when are these equal? Or, phrased in the setting of the de Rham complex on a manifold M, is every closed differential form exact? More explicitly, if ω is an arbitrary closed q-form, is there always a $(q-1)$-form ϕ such that $d\phi = \omega$? We measure our failure to solve for ϕ on M by the construction of the *de Rham Cohomology.*

Definition 2.6: *The q-th* de Rham cohomology *of a manifold M is the quotient space*

$$\begin{aligned} H^q_{dR}(M) &= \{\textit{closed } q\textit{-forms}\}/\{\textit{exact } q\textit{-forms}\} \\ &= \ker(d : \mathbf{\Omega^q}(M) \to \mathbf{\Omega^{q+1}}(M))/\operatorname{im}(d : \mathbf{\Omega^{q-1}}(M) \to \mathbf{\Omega^q}(M)). \end{aligned}$$

For notational simplicity we will denote $d : \mathbf{\Omega^q}(M) \to \mathbf{\Omega^{q+1}}(M)$ by d_q and an equivalence class of elements in $H^q_{dR}(M)$ by $[\omega]$ where all of the elements in $[\omega]$ are of the form $\omega + d\phi$ for some $\phi \in \mathbf{\Omega^{q-1}}(M)$. More generally, we can define the cohomology of a given complex C.

Definition 2.7: *The q-th* cohomology *of a complex C is the quotient space*

$$H^q(C) = \ker(\partial_i)/\operatorname{im}(\partial_{i-1}).$$

In general, the cohomology *of a complex C is the direct sum of these quotient spaces:*

$$H^*(C) = \bigoplus_{q \in \mathbb{Z}} H^q(C).$$

Proposition 2.3: *The dimension of $H^0_{dR}(M)$ is precisely the number of connected components of an n-dimensional manifold M.*

Proof. Firstly, we note that

$$\begin{aligned} H^0_{dR}(M) &= \ker(d_0)/\mathrm{im}(d_{-1}) \\ &= \ker(d)/\{0\} \\ &= \ker(d) \end{aligned}$$

Since 0-forms are smooth functions, we are looking for f such that

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i = 0 \implies \frac{\partial f}{\partial x_i} = 0 \text{ for all } i = 1, ..., n$$

These are exactly the locally constant functions on M - or specifically, functions that are constant on each connected component of M. Denote each connected component U_i and let $f_i : M \to \mathbb{R}$ be the function

$$f_i(x) = \begin{cases} 1 & \text{if } x \in U_i \\ 0 & \text{else} \end{cases}$$

We can then write any function in $H^0_{dR}(M)$ as $f = \sum_i c_i f_i$ for some $c_i \in \mathbb{R}$. □

Corollary 2.4: *If a manifold M has n connected components then $H^0_{dR}(M) \cong \mathbb{R}^n$.*

Explicitly trying to find $H^q_{dR}(M)$ will prove too difficult for a general manifold to begin with, so we will attempt to find the cohomology of a familiar manifold, the real line.

Example 2.3: We will show that $H^q_{dR}(\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$

Firstly, we know that any two points $a < b \in \mathbb{R}$ can be joined by a line segment $[a, b] \subset \mathbb{R}$ so clearly $\mathbb{R}$ is connected and $H^0_{dR}(\mathbb{R}) = \mathbb{R}$. Looking at higher dimensions, we note that the elements of $\boldsymbol{\Omega^q}(\mathbb{R})$ are of the form

$$\omega = \sum f\, dx_{i_1} \wedge \cdots \wedge dx_{i_q},$$

however $\mathbb{R}$ only has one element in its standard basis, say x. Therefore, since $dx \wedge dx = 0$, we must have that $\boldsymbol{\Omega^q}(\mathbb{R}) = 0$ for all $q > 1$ and thus, $H^q_{dR}(\mathbb{R}) = 0$ for all $q > 2$. Now let

$$\omega = f\, dx \in \boldsymbol{\Omega^1}(\mathbb{R})$$

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and define $F \in \mathbf{\Omega^0}(\mathbb{R})$ by

$$F(x) = \int_0^x f(t)\,dt$$

By the fundamental theorem of calculus,

$$\begin{aligned} dF &= d\left(\int_0^x f(t)\,dt\right) \\ &= \frac{d}{dx}\left(\int_0^x f(t)\,dt\right)\,dx \\ &= f(x)\,dx \\ &= \omega \end{aligned}$$

Since $\mathrm{im}(d_0) = \mathbf{\Omega^1}(\mathbb{R})$ and $\ker(d_0) = \mathbf{\Omega^1}(\mathbb{R})$, $H^1_{dR}(\mathbb{R}) = \ker(d_1)/\mathrm{im}(d_0)$ – and thus

$$H^q_{dR}(\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

3 The Mayer-Vietoris Sequence

Rather than calculating the cohomology of spaces on a case by case basis we would like to take results that are simple to prove (e.g. the cohomology of the real line) and relate these results to more complex spaces. In this section we will first show that if a manifold is *contractible* then it will be endowed with a particular cohomology and then use the cohomologies of *contractible* manifolds as the building blocks of cohomologies of more interesting manifolds.

3.1 Pullbacks

Suppose we would like to define a smooth function $g : \mathbb{R}^n \to \mathbb{R}$ on a different domain $U \subset \mathbb{R}^m$. If we have a function $f : U \to \mathbb{R}^n$ then by defining

$$f^*(g) = g \circ f : U \to \mathbb{R}$$

we can effectively 'pull' the domain of g back to U. In order to preserve structure, we seek a definition of f^* on differential forms that commutes with d.

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Definition 3.1: *Let f be a smooth function $f : \mathbb{R}^m \to \mathbb{R}^n$, $f(x_1, ..., x_m) = (f_1, ..., f_n)$ where $f_i : \mathbb{R}^m \to \mathbb{R}$. Then f induces a pullback map $f^* : \mathbf{\Omega^*}(\mathbb{R}^n) \to \mathbf{\Omega^*}(\mathbb{R}^m)$ on a form $\omega = \sum g_{i_1 \cdots i_q}\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}$ by*

$$f^*(\omega) = f^*\left(\sum g_{i_1 \cdots i_q}\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}\right) = \sum (g_{i_1 \cdots i_q} \circ f)\, df_{i_1} \wedge \cdots \wedge df_{i_q}$$

Example 3.1: First, for notational simplicity, denote the standard basis of $\mathbb{R}^2$ by $\{u, v\}$ and the standard basis of $\mathbb{R}^3$ by $\{x, y, z\}$. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (y\, sin(x), z)$ and $\omega = u^2 v\, du \wedge dv \in \mathbf{\Omega^2}(\mathbb{R}^2)$.

$$\begin{aligned} f^*(\omega) =& f^*(u^2 v\, du \wedge dv) \\ =& (y\, sin(x))^2 z\, d(y\, sin(x)) \wedge d(z) \\ =& y^2 sin^2(x)\, z\, (sin(x)\, dy + y\, cos(x)\, dx) \wedge dz \\ =& y^2 sin^3(x)\, z\, dy \wedge dx + y^3 sin^2(x) cos(x)\, z\, dx \wedge dz \end{aligned}$$

For a proof that d commutes with f^* we refer the reader to the proof presented in Bott & Tu [1, p.19]. If we look back at example 1.1 and example 1.2 we notice these are just pullbacks of differential forms defined on a curve to the unit interval. We will use the idea of a pullback of a differential form to find the cohomology of all manifolds that are *contractible.*

Definition 3.2: *A topological space X is* contractible *if the identity map* $\mathrm{id}_X : X \to X$ *is homotopic to a constant map. That is, there is a continuous map $h : X \times [0, 1] \to X$ such that*

$$h(x, 0) = x_0 \textit{ and } h(x, 1) = \mathrm{id}_X(x) = x$$

To prove our next big theorem we are going to use a homotopy as a pullback on a differential form. As defined above, this will require the map $h : X \times [0, 1] \to X$ (sometimes called a *contraction* of X) to be smooth which is not a condition imposed in the definition of a manifold being contractible. As it turns out, we can always find a smooth contraction if there exists a continuous contraction and for those interested in the proof we direct the reader to proposition 17.8 in Bott & Tu [1, p.213].

Theorem 3.1 (Poincaré Lemma)**:** *Suppose a manifold M is contractible. Then*

$$H^q_{dR}(M) \cong \begin{cases} \mathbb{R} & \textit{if } q = 0 \\ 0 & \textit{else} \end{cases}$$

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Proof. Suppose M is contractible with a smooth contraction $h : M \times [0,1] \to M$ where $h(x,1) = \mathrm{id}_M(x)$ and $h(x,0) = x_0$ for some $x_0 \in M$. The method of proof will involve showing that for $q > 1$ every closed q-form is exact and hence $\ker(d_q) = \mathrm{im}(d_{q-1}) \implies H^q_{dR}(M) = 0$. Given a form $\omega \in \boldsymbol{\Omega^q}(M)$,

$$\begin{aligned} h^*(\omega) =& h^*\left(\sum f_{i_1\cdots i_q}\, dx_{i_1} \wedge \cdots \wedge dx_{i_q}\right) \\ =& \sum f_{i_1\cdots i_q} \circ h\, dh_{i_1} \wedge \cdots \wedge dh_{i_q} \end{aligned}$$

which, because h is defined on $M \times [0,1]$, will then be of the form

$$\begin{aligned} h^*(\omega) =& \sum g_{i_1\cdots i_q}\, dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ &+ \sum k_{j_1\cdots j_{q-1}}\, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{q-1}} \end{aligned}$$

where $t \in [0,1]$ and $g_{i_1\cdots 1_q}, k_{j_1\cdots j_{q-1}} \in \boldsymbol{\Omega^0}(M \times [0,1])$. By defining

$$K\omega = \sum \left(\int_0^1 k_{j_1\cdots j_{q-1}}\, dt\right) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{q-1}}$$

We are turning $\omega \in \boldsymbol{\Omega^q}(M)$ into a $(q-1)$-form. Since

$$\begin{aligned} h^*(d\omega) = dh^*(\omega) =& \sum \frac{\partial g_{i_1\cdots i_q}}{\partial x_i}\, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ &+ \sum \frac{\partial g_{i_1\cdots i_q}}{\partial t}\, dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ &+ \sum \frac{\partial k_{j_1\cdots j_{q-1}}}{\partial x_i}\, dx_i \wedge dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{q-1}} \end{aligned}$$

we observe the relationship

$$\begin{aligned} dK\omega + Kd\omega =& \sum \left(\int_0^1 \frac{\partial k_{j_1\cdots j_{q-1}}}{\partial x_i} dt\right) \wedge dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{q-1}} \\ &+ \sum \left(\int_0^1 \frac{\partial g_{i_1\cdots i_q}}{\partial t} dt\right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \\ &- \sum \left(\int_0^1 \frac{\partial k_{j_1\cdots j_{q-1}}}{\partial x_i} dt\right) \wedge dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{q-1}} \\ =& \sum \left(\int_0^1 \frac{\partial g_{i_1\cdots i_q}}{\partial t} dt\right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \end{aligned}$$

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But since

$$\frac{\partial}{\partial t}(dt \wedge h^*(\omega)) = \sum \frac{\partial g_{i_1 \cdots i_q}}{\partial t} dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

and by the fundamental theorem of calculus,

$$dK\omega + Kd\omega = h^*(\omega)\big|_{t=1} - h^*(\omega)\big|_{t=0}$$

But we know that $h^*(\omega)\big|_{t=1} = \mathrm{id}_M(\omega) = \omega$ and $h^*(\omega)\big|_{t=0} = 0$ since all the derivatives of a constant map vanish. Thus we have that

$$dK\omega + Kd\omega = \omega$$

For a closed differential form ω we thus have that

$$dK\omega + Kd\omega = dK\omega + K(0) = \omega$$

and hence ω is the derivative of the differential $(p-1)$-form $K\omega$ and hence every closed form is exact. From this, we can deduce $H^q_{dR}(M) = 0$ for all $q > 0$. For $q = 0$ we realise that since M is contractible to a point $x_0 \in M$, everything is path connected to x_0 and hence M is connected, implying that $H^0_{dR}(M) = \mathbb{R}$. Thus,

$$H^q_{dR}(M) \cong \begin{cases} \mathbb{R} & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

□

With this result, we have categorised the cohomology of every contractible manifold. In order to extend this to more exotic manifolds we will have to think of a more sophisticated technique.

3.2 Exact Sequences

Earlier we introduced the notion of a differential complex. A differential complex or, more generally, any sequence of vector spaces

$$\cdots \longrightarrow C^{q-1} \xrightarrow{\partial_{q-1}} C^q \xrightarrow{\partial_q} C^{q+1} \longrightarrow \cdots$$

is called *exact* if $\ker(\partial_q) = \mathrm{im}(\partial_{d_{q-1}})$ for all $q \in \mathbb{Z}$. As we have seen with the Poincaré Lemma above, the sequence

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$$\cdots \longrightarrow \boldsymbol{\Omega^{q-1}}(M) \xrightarrow{d} \boldsymbol{\Omega^{q}}(M) \xrightarrow{d} \boldsymbol{\Omega^{q+1}}(M) \longrightarrow \cdots$$

is exact. We could also specify a special kind of exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

which we call a *short exact sequence.* The reason it is special is because we have forced the linear maps f and g to be injective and surjective respectively.

Theorem 3.2: *Consider the short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Then f is injective and g is surjective.

Proof. By definition f is injective if $\ker(f) = 0$. Since the map from $0 \to A$ must be a linear map, we must have that its image is 0. Thus, by exactness

$$\mathrm{im}(0 \to A) = 0 = \ker(f)$$

We also have that the kernel of the map from $C \to 0$ must be all of C. Once again, by exactness

$$\ker(C \to 0) = C = \mathrm{im}(g),$$

so g is surjective by definition. □

We could also have the scenario where A, B and C are complexes themselves. If the maps f and g are defined such that they commute with the differential operators of each complex – i.e. for

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & d_A \uparrow & & d_B \uparrow & & d_C \uparrow & & \\
0 & \longrightarrow & A^{q+1} & \xrightarrow{f} & B^{q+1} & \xrightarrow{g} & C^{q+1} & \longrightarrow & 0 \\
 & & d_A \uparrow & & d_B \uparrow & & d_C \uparrow & & \\
0 & \longrightarrow & A^{q} & \xrightarrow{f} & B^{q} & \xrightarrow{g} & C^{q} & \longrightarrow & 0 \\
 & & d_A \uparrow & & d_B \uparrow & & d_C \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

we have that $f \circ d_A = d_B \circ f$ and $g \circ d_B = d_C \circ g$ – we call f and g chain maps. This induces a result with the cohomologies of A, B and C that is important in constructing the cohomology of non-contractible manifolds.

Remark: Note that we say that a sequence of differential complexes

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is short exact if the sequence

$$0 \longrightarrow A^q \xrightarrow{f} B^q \xrightarrow{g} C^q \longrightarrow 0$$

is a short exact sequence for all $q \in \mathbb{Z}$

Lemma 3.3: *Given a short exact sequence of differential complexes*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in which f and g are chain maps, the natural maps f^* and g^* where

$$H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C)$$

are well defined and form an exact sequence.

Proof. Firstly, for $a \in A^q$ closed, denote $[a] \in H^q(A)$ to be the coset of closed q-forms containing a (i.e. $a' \in [a] \iff a' - a$ is exact). We want the functions $f^* : H^q(A) \to H^q(B)$ and $g^* : H^q(B) \to H^q(C)$ where $f^*([a]) = [f(a)]$ and $g^*([b]) = [g(b)]$ to be well defined. Suppose $a' - a$ is exact in A^q, so then $[a'] = [a]$. We can therefore write $a' = a + d_A\omega$ for $\omega \in A^{q-1}$. Thus

$$\begin{aligned} f(a') &= f(a + d_A\omega) \\ &= f(a) + f(d_A\omega) \text{ [by linearity]} \\ &= f(a) + d_B f(\omega) \text{ [since } f \text{ is a chain map]} \end{aligned}$$

So we then have $f(a') - f(a) = d_B f(\omega)$ and therefore $f^*([a]) = [f(a)] = [f(a')] = f^*([a'])$. Notice that since we only used the linearity of f and the fact it is a chain map, thus the method is analogous for showing g^* is well defined.

Now we want to show that $\ker(g^*) = \text{im}(f^*)$. Suppose $[b] \in \ker(g^*)$, implying that

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$g^*([b]) = [0] \in H^q(C)$. Therefore $g(b)$ is exact $\implies$ $g(b) = d_B c$ for some $c \in C^{q-1}$. But g is surjective, so $c = g(b')$ for some $b' \in B^{q-1}$.

$$\therefore\ g(b) = d_C g(b')$$
$$\therefore\ g(b) = g(d_B b')$$
$$\therefore\ g(b - d_B b') = 0$$

Then there is an $a \in A^q$ such that $f(a) = b - d_B b'$ since $\ker(g) = im(f)$.
Note too that $f(d_A a) = d_B f(a) = d_B(b - d_B b') = d_B b - d_B^2 b' = 0$ since b is closed and $d_B^2 = 0$. So, by the injectivity of f, $d_A a = 0$ which implies a is closed. Therefore $[b] \in \ker(g^*)$ implies that there is an $[a] \in H^q(A)$ such that $f^*([a]) = [b]$ and thus

$$\ker(g^*) \subset \mathrm{im}(f^*).$$

Now suppose $[b] \in \mathrm{im}(f^*)$, i.e. that $[b] = f^*([a])$ for some $[a] \in H^q(A)$. Then let $a \in [a]$ where $f(a) = b' \in [b]$. Since $\ker(g) = \mathrm{im}(f)$, we have that $g(f(a)) = 0$. But then $g^*([b]) = g^*([b']) = [g(b')] = [0]$. Therefore $[b] \in \mathrm{im}(f^*)$ which implies $g^*([b]) = [0]$. So

$$\ker(g^*) \supset \mathrm{im}(f^*) \implies \ker(g^*) = \mathrm{im}(f^*)$$

□

This gives us an important result about cohomologies, but we are one step away from an even more powerful result. The next result, the *snake lemma*, relates the q^{th} cohomology of C to the $(q+1)^{th}$ cohomology of A.

Theorem 3.4 (Snake Lemma)**:** *Given a short exact sequence of differential complexes*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in which f and g are chain maps, there is a homomorphism $\partial : H^q(C) \to H^{q+1}(A)$ such that the sequence in the following diagram is exact.

$$\cdots \xrightarrow{\partial} H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C) \xrightarrow{\partial} H^{q+1}(A) \xrightarrow{f^*} \cdots$$

The proof of this result is slightly more technical than the result above and hence is included only as an appendix for those interested. We can utilise this sequence by splitting a manifold M into two subsets U and V and form the *Mayer-Vietoris sequence*. We will first have to take a slight detour to define a *partition of unity*.

Definition 3.3: *A* partition of unity *on an open cover* $\{U_\alpha\}_{\alpha\in I}$ *of a manifold* M *is a collection of non-negative smooth functions* $\{\rho_\alpha\}_{\alpha\in I}$ *such that*

1. *Every point has a neighbourhood in which* $\sum_{\alpha\in I}\rho_\alpha$ *is a finite sum*
2. $\sum_{\alpha\in I}\rho_\alpha = 1$
3. $\rho_\alpha(x) = 0$ *for all* $x \notin U_\alpha$

It is a fact that every cover of a manifold can be endowed with a partition of unity. Unfortunately the proof is well beyond the scope of this project and we direct the curious reader to Madsen & Tornehave [2, p.221-225]. Now, onto the main event!

Theorem 3.5: *Let* M *be a manifold with* $U, V \subset M$ *manifolds such that* $U \cup V = M$. *The* Mayer-Vietoris sequence,

$$0 \longrightarrow \mathbf{\Omega}^*(M) \xrightarrow{\ i\ } \mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V) \xrightarrow{\ j\ } \mathbf{\Omega}^*(U\cap V) \longrightarrow 0$$

where $i : \mathbf{\Omega}^*(M) \to \mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V)$ *is defined by*

$$i(\omega) = (\omega\big|_U, \omega\big|_V)$$

and $j : \mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V) \to \mathbf{\Omega}^*(U\cap V)$ *is defined by*

$$j(\omega_1, \omega_2) = \omega_2\big|_{U\cap V} - \omega_1\big|_{U\cap V}$$

is an exact sequence.

Proof. Firstly, we would like to show that $\ker(i) = 0$. This is simple to show since

$$i(\omega) = (\omega\big|_U, \omega\big|_V) = (0, 0)$$

if and only if $\omega = 0$ globally. We can also see that an element of the kernel of j, i.e. an element (ω_1, ω_2) such that

$$j(\omega_1, \omega_2) = \omega_2\big|_{U\cap V} - \omega_1\big|_{U\cap V} = 0,$$

must have that each component agrees on the overlap $U \cap V$ and therefore is a global form in the image of i. Showing j is surjective will require the use of a partition of unity $\{\rho_U, \rho_V\}$ on $\{U, V\}$, which covers M. Firstly, given $\omega \in \mathbf{\Omega}^*(U\cap V)$ we can define an object $\omega' \in \mathbf{\Omega}^*(M)$ by

$$\omega'(x) = \begin{cases} \omega & \text{if } x \in U\cap V \\ 0 & \text{else} \end{cases}$$

This is not a differential form because ω' is not necessarily smooth along the boundary of $U \cap V$. However, the forms

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$$\rho_V \omega'\big|_U \in \mathbf{\Omega}^*(U) \text{ and } \rho_U \omega'\big|_V \in \mathbf{\Omega}^*(V)$$

are smooth on their respective sets. From this construction we have

$$\begin{aligned} j(-\rho_V\omega', \rho_U\omega') =& \rho_U\omega'\big|_{U\cap V} - (-\rho_V\omega')\big|_{U\cap V} \\ =& \omega \end{aligned}$$

and since ω was arbitrary, j is surjective. Thus the Mayer-Vietoris sequence is exact. $\square$

From the Mayer-Vietoris sequence, as shown in the Snake Lemma, we get the long exact sequence of cohomologies.

$$\begin{array}{ccccccccc}
 & \longrightarrow & H^2_{dR}(M) & \xrightarrow{i^*} & H^2((\mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V)) & \xrightarrow{j^*} & \cdots & & \\
 & & & & \partial & & & & \\
 & \longrightarrow & H^1_{dR}(M) & \xrightarrow{i^*} & H^1((\mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V)) & \xrightarrow{j^*} & H^1_{dR}(U \cap V) & - & \\
 & & & & \partial & & & & \\
0 & \longrightarrow & H^0_{dR}(M) & \xrightarrow{i^*} & H^0((\mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V)) & \xrightarrow{j^*} & H^0_{dR}(U \cap V) & - &
\end{array}$$

We will use this to calculate the cohomology of the circle, which as we will see is not contractible.

Example 3.2: Consider the circle S^1 with cover $\{U, V\}$ where U is S^1 with its north pole removed and V is S^1 with its south pole removed. From these two sets, we construct the Mayer-Vietoris sequence of the circle:

$$0 \longrightarrow \mathbf{\Omega}^*(S^1) \xrightarrow{i} \mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V) \xrightarrow{j} \mathbf{\Omega}^*(U \cap V) \longrightarrow 0$$

in order to induce the long exact sequence of cohomologies:

$$0 \to H^0_{dR}(S^1) \xrightarrow{i^*} H^0(\mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V)) \xrightarrow{j^*} H^0_{dR}(U \cap V) \xrightarrow{\partial} H^1_{dR}(S^1) \to \cdots$$

Before proceeding, we will need to make sense of the term

$$H^0(\mathbf{\Omega}^*(U) \oplus \mathbf{\Omega}^*(V)).$$

In fact, we find that for any complexes A^* and B^*

$$H^q(A^* \oplus B^*) = H^q(A^*) \oplus H^q(B^*)$$

due to the fact that

$$\begin{aligned} \ker(d_{A\oplus B}) &= \ker(d_A) \oplus \ker(d_B) \\ \operatorname{im}(d_{A\oplus B}) &= \operatorname{im}(d_A) \oplus \operatorname{im}(d_B) \end{aligned}$$

So we in fact have the long exact sequence:

$$0 \to H^0_{dR}(S^1) \xrightarrow{i^*} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{j^*} H^0_{dR}(U \cap V) \xrightarrow{\partial} H^1_{dR}(S^1) \to \cdots$$

From the Poincaré Lemma we know that

$$H^q_{dR}(U) \cong H^q_{dR}(V) \cong \begin{cases} \mathbb{R} & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

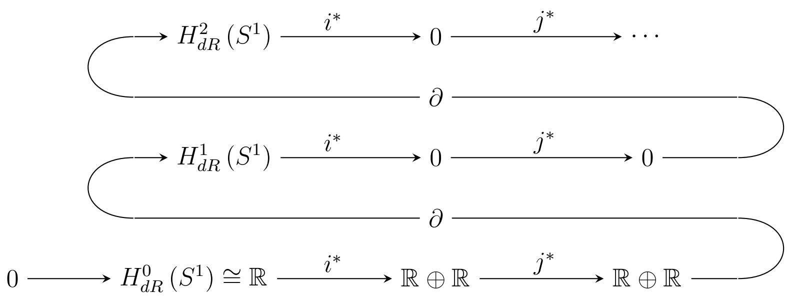
And because $U \cap V$ is S^1 with both poles removed, we could write it as the union of two disjoint sets, say L and R for the left and right half respectively. Using the map i (from the Mayer-Vietoris sequence) we find that

$$\boldsymbol{\Omega}^{\boldsymbol{q}}(L \cup R) \xrightarrow{i} \boldsymbol{\Omega}^{\boldsymbol{q}}(L) \oplus \boldsymbol{\Omega}^{\boldsymbol{q}}(R)$$

is an isomorphism. We therefore have that

$$H^q_{dR}(U \cap V) \cong H^q_{dR}(U) \oplus H^q_{dR}(V) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

We now know enough to calculate the cohomology of the circle using only algebra. From proposition 1.3 and the fact that S^1 is connected we know that $H^0_{dR}(S^1) \cong \mathbb{R}$. We now consider what we know summarised in the following diagram.



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To find $H^1_{dR}(S^1)$ we note that since $\ker(i^*) = H^1_{dR}(S^1)$, by exactness we just need to find $\text{im}(\partial)$. With some sneaky algebra using exactness we find that

$$\begin{aligned} H^1_{dR}(S^1) = \text{im}(\partial) &= \mathbb{R} \oplus \mathbb{R} / \ker(\partial) \\ &= \mathbb{R} \oplus \mathbb{R} / \text{im}(j^*) \end{aligned}$$

But since i^* is injective and

$$\begin{aligned} \text{im}(j^*) &= \mathbb{R} \oplus \mathbb{R} / \ker(j^*) \\ &= \mathbb{R} \oplus \mathbb{R} / \text{im}(i^*) \cong \mathbb{R} \end{aligned}$$

we have that

$$H^1_{dR}(S^1) = \mathbb{R} \oplus \mathbb{R} / \mathbb{R} \cong \mathbb{R}$$

Note that since for all $q > 1$, $H^q_{dR}(U \cap V) = H^q_{dR}(U) \oplus H^q_{dR}(V) = 0$ we get that

$$\partial : 0 \to H^q_{dR}(S^1)$$

is an isomorphism by exactness. With all this information, we have calculated the cohomology of S^1,

$$H^q_{dR}(S^1) \cong \begin{cases} \mathbb{R} & \text{if } q = 0 \\ \mathbb{R} & \text{if } q = 1 \\ 0 & \text{else} \end{cases}$$

3.3 Generalising the Mayer-Vietoris Sequence

Although the Mayer-Vietoris sequence is a very useful tool for computing the cohomology of a manifold, we wish to extend this idea to a more general setting. With our understanding thus far, we are required to find a cover of a manifold M, $\{U, V\}$, such that we are able to make deductions on the cohomology of M using the long exact sequence:

$$\cdots \to H^{q-1}_{dR}(U \cap V) \to H^q_{dR}(M) \to H^q_{dR}(U) \oplus H^q_{dR}(V) \to H^q_{dR}(U \cap V) \to H^{q+1}_{dR}(M) \to \cdots$$

This relies on us being able to calculate $H^q_{dR}(U) \oplus H^q_{dR}(V)$ and $H^q_{dR}(U \cap V)$, which may not be as easy as our example with the circle.

Using the Poincaré Lemma, we are easily able to deduce the cohomology of a contractible set — a set that is homotopy equivalent to a point — so it would be useful if we could somehow restrict the sets we cover our manifold with to contractible sets such that their intersections are also contractible. This is the notion of a *good cover*.

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Definition 3.4: *Let M be a manifold of dimension n. An open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M is called a good cover if all nonempty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are contractible*

This is complemented well by the following theorem from Bott & Tu [1, p.42]

Theorem 3.6: *Every manifold has a good cover. If the manifold is compact, then the cover may be chosen to be finite.*

The Mayer-Vietoris sequence will only be useful here if we can reduce our cover to two sets, so we now consider a more general complex to accommodate a more general covering.

4 Construction of the Čech Complex

4.1 Construction of the difference operator δ through a motivating example

We will attempt to first generalise the Mayer-Vietoris to a three-set cover before generalising even further to a cover of countable sets.

Consider a manifold M with a good cover $\mathfrak{U} = \{U_1, U_2, U_3\}$ and the sequence:

$$\mathbf{\Omega}^*(M) \xrightarrow{\delta} \prod_{i=1}^{3} \mathbf{\Omega}^*(U_i) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \prod_{\substack{i,j=1\\ i<j}}^{3} \mathbf{\Omega}^*(U_i \cap U_j) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}}} \mathbf{\Omega}^*(U_1 \cap U_2 \cap U_3) \to 0 \to 0 \to \cdots$$

where the δ_i operators are restrictions — we will explicitly define them shortly. First, let's think about how elements of the sets in our sequence look.

We are familiar with $\mathbf{\Omega}^*(M)$ which is just composed of the global forms on M, so we start by thinking about elements of $\prod_{i=1}^{3} \mathbf{\Omega}^*(U_i)$. We can think of these elements as vectors with three components — i.e. $\zeta = (\omega_1, \omega_2, \omega_3)$ where $\omega_i \in \mathbf{\Omega}^*(U_i)$ — and define the operator $\delta : \mathbf{\Omega}^*(M) \to \prod_{i=1}^{3} \mathbf{\Omega}^*(U_i)$ by:

$$\delta\omega = (\omega\big|_{U_1}, \omega\big|_{U_2}, \omega\big|_{U_3})$$

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Elements of $\prod\limits_{\substack{i,j=1\\i<j}}^{3} \mathbf{\Omega}^*(U_i \cap U_j)$ can then be thought of a a vector with $\binom{3}{2}$ components, i.e. $\xi = (\omega_{12}, \omega_{13}, \omega_{23})$ where $\omega_{ij} \in \mathbf{\Omega}^*(U_i \cap U_j)$. In this example, we are only left with three sets, but one might imagine how large $\binom{n}{2}$ can get when we generalise to a cover of n sets — or even infinite sets! We define the operators $\delta_0, \delta_1 : \prod\limits_{i=1}^{3} \mathbf{\Omega}^*(U_i) \to \prod\limits_{\substack{i,j=1\\i<j}}^{3} \mathbf{\Omega}^*(U_i \cap U_j)$ by:

$$\delta_0(\omega_1, \omega_2, \omega_3) = (\omega_2\big|_{U_1}, \omega_3\big|_{U_1}, \omega_3\big|_{U_2})$$
$$\delta_1(\omega_1, \omega_2, \omega_3) = (\omega_1\big|_{U_2}, \omega_1\big|_{U_3}, \omega_2\big|_{U_3})$$

We then define an operator $\delta : \prod\limits_{i=1}^{3} \mathbf{\Omega}^*(U_i) \to \prod\limits_{\substack{i,j=1\\i<j}}^{3} \mathbf{\Omega}^*(U_i \cap U_j)$ by:

$$\delta = \delta_0 - \delta_1$$
$$\delta(\omega_1, \omega_2, \omega_3) = (\omega_2\big|_{U_1} - \omega_1\big|_{U_2}, \omega_3\big|_{U_1} - \omega_1\big|_{U_3}, \omega_3\big|_{U_2} - \omega_2\big|_{U_3})$$

It is certainly important to note here that $\delta^2\omega = 0$ if $\omega \in \mathbf{\Omega}^*(M)$ since

$$\omega_i\big|_{U_j} - \omega_j\big|_{U_i} = \omega\big|_{U_i}\Big|_{U_j} - \omega\big|_{U_j}\Big|_{U_i} = \omega\big|_{U_i \cap U_j} - \omega\big|_{U_i \cap U_j} = 0$$

so we are part way to making a differential complex.

We now consider elements of our final set, $\mathbf{\Omega}^*(U_1 \cap U_2 \cap U_3)$. We are now interested in defining the δ_i operators to complete out differential complex. We define $\delta_0, \delta_1, \delta_2 : \prod\limits_{\substack{i,j=1\\i<j}}^{3} \mathbf{\Omega}^*(U_i \cap U_j) \to \mathbf{\Omega}^*(U_1 \cap U_2 \cap U_3)$ by:

$$\delta_0(\omega_{12}, \omega_{13}, \omega_{23}) = \omega_{23}\big|_{U_1}$$
$$\delta_1(\omega_{12}, \omega_{13}, \omega_{23}) = \omega_{13}\big|_{U_2}$$
$$\delta_2(\omega_{12}, \omega_{13}, \omega_{23}) = \omega_{12}\big|_{U_3}$$

and in order to extend this to a differential operator, we define

$$\delta : \prod_{\substack{i,j=1\\i<j}}^{3} \mathbf{\Omega}^*(U_i \cap U_j) \to \mathbf{\Omega}^*(U_1 \cap U_2 \cap U_3) \text{ by:}$$
$$\delta = \delta_0 - \delta_1 + \delta_2,$$
$$\delta(\omega_{12}, \omega_{13}, \omega_{23}) = \omega_{23}\big|_{U_1} - \omega_{13}\big|_{U_2} + \omega_{12}\big|_{U_3}$$

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We can see that this is a differential operator since:

$$\begin{aligned}
\delta^2(\omega_1,\omega_2,\omega_3) &= (\omega_2\big|_{U_1} - \omega_1\big|_{U_2}, \omega_3\big|_{U_1} - \omega_1\big|_{U_3}, \omega_3\big|_{U_2} - \omega_2\big|_{U_3}) \\
&= (\omega_3\big|_{U_2} - \omega_2\big|_{U_3})\Big|_{U_1} - (\omega_3\big|_{U_1} - \omega_1\big|_{U_3})\Big|_{U_2} + (\omega_3\big|_{U_2} - \omega_2\big|_{U_3})\Big|_{U_3} \\
&= (\omega_3\big|_{U_1\cap U_2} - \omega_3\big|_{U_2\cap U_1}) - (\omega_2\big|_{U_3\cap U_1} - \omega_2\big|_{U_1\cap U_3}) + (\omega_1\big|_{U_2\cap U_3} - \omega_1\big|_{U_3\cap U_2}) \\
&= 0
\end{aligned}$$

Thus, we have defined a differential complex on a manifold with a three-set covering with differential operator δ.

4.2 Generalising to a countable collection of sets

Suppose we now have a manifold M with a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$. The condition we will impose on $\mathfrak{U}$ is that A, the index set, is countable and ordered. Since the set is countable, the notion of ordering may seem trivial - but it will be much easier to make sense of the complex with a clear notion of ordering. We will also simplify our notation by denoting $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ by $U_{\alpha_0\alpha_1\cdots\alpha_p}$ and by letting $\omega_{\alpha_0\cdots\alpha_p}$ denote a form on $U_{\alpha_0\cdots\alpha_p}$.
Much as before, we will consider a more general sequence:

$$\mathbf{\Omega}^*(M) \xrightarrow{\delta} \prod_{\alpha\in A} \mathbf{\Omega}^*(U_\alpha) \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \prod_{\substack{\alpha,\beta\in A\\ \alpha<\beta}} \mathbf{\Omega}^*(U_{\alpha\beta}) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}}}\!\!\!\!\rightarrow \prod_{\substack{\alpha,\beta,\gamma\in A\\ \alpha<\beta<\gamma}} \mathbf{\Omega}^*(U_{\alpha\beta\gamma}) \rightrightarrows\!\!\!\!\rightrightarrows \cdots$$

The δ_i operators will act just as they did in the above example, but for clarity we will try to define them explicitly in this generalised context. Suppose:

$$\delta_i : \prod_{\substack{\alpha_0,\dots,\alpha_p\in A\\ \alpha_0<\alpha_1<\cdots<\alpha_p}} \mathbf{\Omega}^*(U_{\alpha_0\cdots\alpha_p}) \to \prod_{\substack{\alpha_0,\dots,\alpha_{p+1}\in A\\ \alpha_0<\cdots<\alpha_{p+1}}} \mathbf{\Omega}^*(U_{\alpha_0\cdots\alpha_{p+1}})$$

is an arbitrary restriction operator defined between p-tuple and $(p+1)$-tuple intersections of sets in $\mathfrak{U}$. Note that every $(p+1)$-tuple intersection must have an i^{th} intersecting set since they are ordered.
So let $\xi \in \prod \mathbf{\Omega}^*(U_{\alpha_0\cdots\alpha_p})$ and consider $(\delta_i\xi)_{\beta_0\cdots\beta_{p+1}}$ - the component of $\delta_i\xi$ pertaining to the intersection $U_{\beta_0} \cap \cdots \cap U_{\beta_{p+1}}$ where $\beta_0 < \cdots < \beta_{p+1}$ and $\beta_0, \dots, \beta_{p+1} \in A$. The operator δ_i (for some $0 \le i \le p+1$) will take the component of ξ that is "missing" the restriction to the set β_i and is already restricted to $U_{\beta_0} \cap \cdots \cap U_{\beta_{i-1}} \cap U_{\beta_{i+1}} \cap \cdots \cap U_{\beta_{p+1}}$. Since this is a specific p-tuple of intersections, this corresponds to a unique component of ξ. More explicitly:

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$$(\delta_i \xi)_{\beta_0 \cdots \beta_{p+1}} = \omega_{\beta_0 \cdots \beta_{i-1}\beta_{i+1}\cdots\beta_{p+1}}\big|_{U_{\beta_i}}$$

We can thus describe the general operator

$$\delta : \prod_{\substack{\alpha_0 < \cdots < \alpha_p \\ \alpha_0, \ldots, \alpha_p \in A}} \mathbf{\Omega}^*(U_{\alpha_0 \cdots \alpha_p}) \to \prod_{\substack{\alpha_0 < \cdots < \alpha_{p+1} \\ \alpha_1, \ldots, \alpha_{p+1} \in A}} \mathbf{\Omega}^*(U_{\alpha_0 \cdots \alpha_{p+1}}) \text{ by:}$$

$$\delta = \sum_{i=0}^{p} (-1)^i \delta_i,$$

$$(\delta \xi)_{\beta_0 \cdots \beta_{p+1}} = \sum_{i=0}^{p} (-1)^i \cdot \omega_{\beta_0 \cdots \beta_{i-1}\beta_{i+1}\cdots\beta_{p+1}}\big|_{U_{\beta_i}}$$

which gives us a well defined expression for every component of $\delta\xi \in \prod \mathbf{\Omega}^*(U_{\alpha_0 \cdots \alpha_{p+1}})$ and thus defines δ uniquely.

Theorem 4.1: $\delta^2 = 0$

Proof. Consider $(\delta^2 \xi)_{\beta_1 \cdots \beta_{p+2}}$

$$\begin{aligned}
(\delta^2 \xi)_{\beta_0 \cdots \beta_{p+2}} &= \sum_{i=0}^{p} (-1)^i \cdot (\delta\xi)_{\beta_0 \cdots \beta_{i-1}\beta_{i+1}\cdots\beta_{p+2}}\big|_{U_{\beta_i}} \\
&= \sum_{i=0}^{p+2} (-1)^i \cdot \Bigg[\sum_{j=0}^{i-1} (-1)^j \omega_{\beta_0 \cdots \beta_{j-1}\beta_{j+1}\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_{p+2}}\big|_{U_{\beta_j}} \\
&\qquad\qquad + \sum_{j=i+1}^{p+2} (-1)^{j-1} \omega_{\beta_0 \cdots \beta_{i-1}\beta_{i+1}\cdots\beta_{j-1}\beta_{j+1}\cdots\beta_{p+2}}\big|_{U_{\beta_j}} \Bigg]\Bigg|_{U_{\beta_i}} \\
&= \sum_{\substack{i,j=0 \\ j<i}}^{p+2} (-1)^{i+j} \omega_{\beta_0 \cdots \beta_{j-1}\beta_{j+1}\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_{p+2}}\big|_{U_{\beta_i} \cap U_{\beta_j}} \\
&\quad + \sum_{\substack{i,j=0 \\ j>i}}^{p+2} (-1)^{i+j-1} \omega_{\beta_0 \cdots \beta_{i-1}\beta_{i+1}\cdots\beta_{j-1}\beta_{j+1}\cdots\beta_{p+2}}\big|_{U_{\beta_i} \cap U_{\beta_j}} \\
&= 0
\end{aligned}$$

Thus $(\delta^2\xi)_{\beta_0 \cdots \beta_{p+2}} = 0$ for every component of $\delta^2\xi$ and for every $\xi \in \prod \mathbf{\Omega}^*(U_{\alpha_0 \cdots \alpha_p})$, and therefore $\delta^2 = 0$. □

We have thus defined a differential complex with differential operator δ and can therefore define a cohomology on this complex. We can also restrict $\mathbf{\Omega}^*$ to $\mathbf{\Omega}^q$ and just consider the sequence:

$$\mathbf{\Omega^q}(M) \xrightarrow{\delta} \prod_{\alpha\in A} \mathbf{\Omega^q}(U_\alpha) \xrightarrow{\delta} \prod_{\substack{\alpha,\beta\in A\\ \alpha<\beta}} \mathbf{\Omega^q}(U_{\alpha\beta}) \xrightarrow{\delta} \prod_{\substack{\alpha,\beta,\gamma\in A\\ \alpha<\beta<\gamma}} \mathbf{\Omega^q}(U_{\alpha\beta\gamma}) \xrightarrow{\delta} \cdots$$

on which we can define an equivalent cohomology. We call the differential complex above a Čech Complex on $\mathfrak{U}$ with values in $\mathbf{\Omega^q}$ which we denote $\check{C}^*(\mathfrak{U}, \mathbf{\Omega^q})$, where $\check{C}^p(\mathfrak{U}, \mathbf{\Omega^q})$ is taken to be the product of all the possible combinations of p-tuple intersections on $\mathfrak{U}$ - or explicitly:

$$\check{C}^p(\mathfrak{U}, \mathbf{\Omega^q}) = \prod_{\substack{\alpha_0,\dots,\alpha_p\in A\\ \alpha_0<\dots<\alpha_p}} \mathbf{\Omega^q}(U_{\alpha_0\cdots\alpha_p})$$

5 Construction of the Čech-de Rham Double Complex

We now have the right ingredients to proceed to a complex that incorporates both the de Rham and the Čech complexes. First, it is helpful to picture the double complex as a lattice:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
\check{C}^0(\mathfrak{U}, \mathbf{\Omega^2}) & \xrightarrow{\delta} & \check{C}^1(\mathfrak{U}, \mathbf{\Omega^2}) & \xrightarrow{\delta} & \check{C}^2(\mathfrak{U}, \mathbf{\Omega^2}) & \xrightarrow{\delta} & \check{C}^3(\mathfrak{U}, \mathbf{\Omega^2}) & \xrightarrow{\delta} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
\check{C}^0(\mathfrak{U}, \mathbf{\Omega^1}) & \xrightarrow{\delta} & \check{C}^1(\mathfrak{U}, \mathbf{\Omega^1}) & \xrightarrow{\delta} & \check{C}^2(\mathfrak{U}, \mathbf{\Omega^1}) & \xrightarrow{\delta} & \check{C}^3(\mathfrak{U}, \mathbf{\Omega^1}) & \xrightarrow{\delta} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
\check{C}^0(\mathfrak{U}, \mathbf{\Omega^0}) & \xrightarrow{\delta} & \check{C}^1(\mathfrak{U}, \mathbf{\Omega^0}) & \xrightarrow{\delta} & \check{C}^2(\mathfrak{U}, \mathbf{\Omega^0}) & \xrightarrow{\delta} & \check{C}^3(\mathfrak{U}, \mathbf{\Omega^0}) & \xrightarrow{\delta} & \cdots
\end{array}$$

where we understand d to operate on an element of $\check{C}^p(\mathfrak{U}, \mathbf{\Omega^q})$ componentwise. It should be noted that the $\mathbf{\Omega^q}(M)$ have been deliberately left out of the large double complex above.

We will also consider a more basic kind of the Čech Complex. Consider for example: $\check{C}^*(\mathfrak{U}, \mathbb{R})$ - the Čech Complex on $\mathfrak{U}$ with values in $\mathbb{R}$. Much like $\check{C}^*(\mathfrak{U}, \mathbf{\Omega^q})$ takes values in q-forms, we can think of this more basic Čech Complex as taking locally constant functions as its values. The astute reader will notice that the locally constant functions on $\check{C}^p(\mathfrak{U}, \mathbf{\Omega^0})$ are exactly the kernel of $d : \check{C}^p(\mathfrak{U}, \mathbf{\Omega^0}) \to \check{C}^p(\mathfrak{U}, \mathbf{\Omega^1})$. We can thus consider the more basic Čech Complex and de Rham Complex being attached to the sides of our bigger Čech-de Rham Complex as follows:

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
\mathbf{\Omega^2}(M) & \overset{\delta}{\dashrightarrow} & \check{C}^0(\mathfrak{U}, \mathbf{\Omega^2}) & \overset{\delta}{\longrightarrow} & \check{C}^1(\mathfrak{U}, \mathbf{\Omega^2}) & \overset{\delta}{\longrightarrow} & \check{C}^2(\mathfrak{U}, \mathbf{\Omega^2}) & \overset{\delta}{\longrightarrow} & \check{C}^3(\mathfrak{U}, \mathbf{\Omega^2}) & \overset{\delta}{\longrightarrow} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
\mathbf{\Omega^1}(M) & \overset{\delta}{\dashrightarrow} & \check{C}^0(\mathfrak{U}, \mathbf{\Omega^1}) & \overset{\delta}{\longrightarrow} & \check{C}^1(\mathfrak{U}, \mathbf{\Omega^1}) & \overset{\delta}{\longrightarrow} & \check{C}^2(\mathfrak{U}, \mathbf{\Omega^1}) & \overset{\delta}{\longrightarrow} & \check{C}^3(\mathfrak{U}, \mathbf{\Omega^1}) & \overset{\delta}{\longrightarrow} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
\mathbf{\Omega^0}(M) & \overset{\delta}{\dashrightarrow} & \check{C}^0(\mathfrak{U}, \mathbf{\Omega^0}) & \overset{\delta}{\longrightarrow} & \check{C}^1(\mathfrak{U}, \mathbf{\Omega^0}) & \overset{\delta}{\longrightarrow} & \check{C}^2(\mathfrak{U}, \mathbf{\Omega^0}) & \overset{\delta}{\longrightarrow} & \check{C}^3(\mathfrak{U}, \mathbf{\Omega^0}) & \overset{\delta}{\longrightarrow} & \cdots \\
 & & \vdots\uparrow i & & \vdots\uparrow i & & \vdots\uparrow i & & \vdots\uparrow i & & \\
 & & \check{C}^0(\mathfrak{U}, \mathbb{R}) & \overset{\delta}{\longrightarrow} & \check{C}^1(\mathfrak{U}, \mathbb{R}) & \overset{\delta}{\longrightarrow} & \check{C}^2(\mathfrak{U}, \mathbb{R}) & \overset{\delta}{\longrightarrow} & \check{C}^3(\mathfrak{U}, \mathbb{R}) & \overset{\delta}{\longrightarrow} & \cdots
\end{array}
$$

For now, we will not dwell on what the function i is specifically.

5.1 The operator D and the Čech-de Rham Cohomology

Our ultimate goal will be to using the double complex above and the maps δ, i pictured to form an isomorphism between the cohomologies of the de Rham Complex on a manifold M (the left most column) and the Čech Complex on $\mathfrak{U}$ with values in $\mathbb{R}$ (the bottom most row). As such, it will be much easier if we can define a single cohomology on our double complex to relate to both the Čech and de Rham cohomologies. We will achieve this by considering the diagonals of our double complex as follows:

Definition 5.1: *Let* $K^n = \bigoplus_{p+q=n} \check{C}^p(\mathfrak{U}, \mathbf{\Omega^q})$

Then we can consider $\phi \in K^n$ as a vector with $n+1$ components:

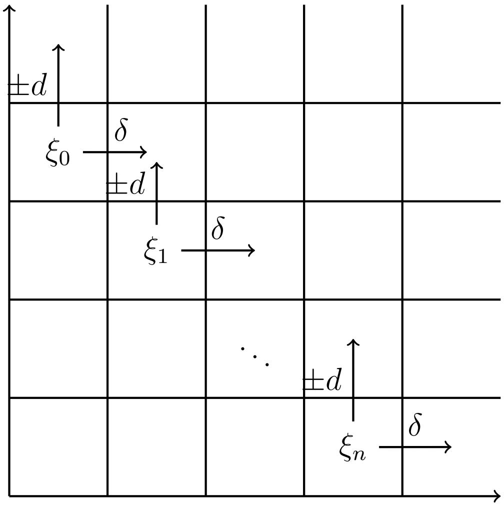
$$\phi = (\xi_0, \xi_1, \ldots, \xi_n) \text{ where } \xi_i \in \check{C}^i(\mathfrak{U}, \mathbf{\Omega^{n-i}})$$

We will define an operator D on this sequence of sets that incorporates both d and δ in such a way that forms a differential complex.

Definition 5.2: *Let $\phi = (\xi_0, \xi_1, \ldots, \xi_n) \in K^n$. We define $D : K^n \to K^{n+1}$ by*

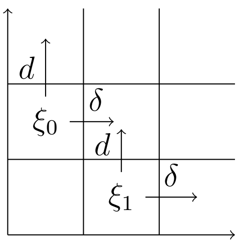
$$(D\phi)_0 = (-1)^n \cdot d\xi_0$$
$$(D\phi)_i = \delta\xi_{i-1} + (-1)^n \cdot d\xi_i \text{ for } 0 < i < n+1$$
$$(D\phi)_{n+1} = \delta\xi_n$$

We can think of D and K^n in a more visual way.



Where $(\xi_0, \ldots, \xi_n)$ is an element of K^n, d is positive if n is even and negative if n is odd.

Example 5.1: Take our previous example where M is a manifold with good cover $\mathfrak{U} = \{U_1, U_2, U_3\}$. Let $\phi = (\xi_0, \xi_1) \in K^1 = \check{C}^0(\mathfrak{U}, \mathbf{\Omega^1}) \bigoplus \check{C}^1(\mathfrak{U}, \mathbf{\Omega^0})$.
Then $\xi_0 = (\omega_1, \omega_2, \omega_3) \in \check{C}^0(\mathfrak{U}, \mathbf{\Omega^1})$ where $\omega_i \in \mathbf{\Omega^1}(U_i)$
and $\xi_1 = (f_{12}, f_{13}, f_{23}) \in \check{C}^1(\mathfrak{U}, \mathbf{\Omega^0})$ where $f_{ij} : U_i \cap U_j \to \mathbb{R}$ is smooth.



Then $D\phi = (d\xi_0, \delta\xi_0 + d\xi_1, \delta\xi_1) \in K^2 = \check{C}^0(\mathfrak{U}, \mathbf{\Omega^2}) \bigoplus \check{C}^1(\mathfrak{U}, \mathbf{\Omega^1}) \bigoplus \check{C}^2(\mathfrak{U}, \mathbf{\Omega^0})$ where:

$$\begin{aligned} d\xi_0 =& (d\omega_1, d\omega_2, d\omega_3) \\ \delta\xi_0 + d\xi_1 =& (\omega_2\big|_{U_1} - \omega_1\big|_{U_2} + df_{12},\ \omega_3\big|_{U_1} - \omega_1\big|_{U_3} + df_{13},\ \omega_3\big|_{U_2} - \omega_2\big|_{U_3} + df_{23}) \\ \delta\xi_1 =& f_{23}\big|_{U_1} - f_{13}\big|_{U_2} + f_{12}\big|_{U_3} \end{aligned}$$

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In order to show that this is a differential operator we need to show that $D^2 = 0$.

Theorem 5.1: *The operators d and δ commute*

We will leave this as an exercise for the reader.

Proposition 5.2: $D^2 = 0$

Proof. Let $\phi \in K^n$ and consider $(D^2\phi)_i$
If $i = 0$ then $(D^2\phi)_0 = (-1)^{n+1} \cdot d(D\phi)_0 = (-1)^{2n+1} \cdot d^2\xi_0 = 0$
Similarly, if $i = n + 2$ then $(D^2\phi)_{n+2} = \delta^2\xi_n = 0$
If $0 < i < n$ then

$$\begin{aligned}(D^2\phi)_i =&\delta(D\phi)_{i-1} + (-1)^{n+1} \cdot d(D\phi)_i \\ =&\delta^2\xi_{i-2} + (-1)^n \cdot \delta d\xi_{i-1} + (-1)^{n+1} \cdot d\delta\xi_{i-1} - d^2\xi_i \\ =&(-1)^n(\delta d\xi_{i-1} - d\delta\xi_{i-1}) \\ =&0\end{aligned}$$

And we therefore have that D is a differential operator on the complex K^* □

Example 5.2 (continued)**:** From the previous example suppose M is 3-dimensional and let $\xi_0 = (-y \cdot dz, z \cdot dx, -x \cdot dy)$, $\xi_1 = (-yz, xy, xz)$. From the previous calculations we have that:

$$\begin{aligned}&D\phi = (\zeta_0, \zeta_1, \zeta_2), \text{ where} \\ &\zeta_0 = d\xi_0 = (-dy \wedge dz, -dx \wedge dz, -dx \wedge dy) \\ &\zeta_1 = \delta\xi_0 + d\xi_1 = (z \cdot dx - z \cdot dy, y \cdot dz + y \cdot dx, x \cdot dz - x \cdot dy) \\ &\zeta_2 = \delta\xi_1 = xz - xy - yz\end{aligned}$$

Then

$$
\begin{aligned}
D^2\phi =&(-d\zeta_0, \delta\zeta_0 - d\zeta_1, \delta\zeta_1 - d\zeta_2, \delta\zeta_2), \text{ where}\\
-d\zeta_0 =& - d(-dy \wedge dz, -dx \wedge dz, -dx \wedge dy)\\
=&(0, 0, 0)\\
\delta\zeta_0 - d\zeta_1 =&(-dx \wedge dz + dy \wedge dz - dz \wedge dx + dz \wedge dy,\\
&- dx \wedge dy + dy \wedge dz - dy \wedge dz - dy \wedge dx,\\
&- dx \wedge dy + dx \wedge dz - dx \wedge dz + dx \wedge dy)\\
=&(0, 0, 0)\\
\delta\zeta_1 - d\zeta_2 =&(x \cdot dz - x \cdot dy - y \cdot dz - y \cdot dx + z \cdot dx - z \cdot dy)\\
&- (z \cdot dx + x \cdot dz - y \cdot dx - x \cdot dy - z \cdot dy - y \cdot dz)\\
=&0\\
\text{and } \delta\zeta_2 =&0 \text{ since } \check{C}^3(\mathfrak{U}, \mathbf{\Omega^0}) = 0 \text{ (i.e. since there are no 4-fold intersections)}
\end{aligned}
$$

So $D^2\phi = 0$ as expected.

6 de Rham's Theorem

In this section we will prove *de Rham's Theorem.*

Theorem 6.1: *Let M be a manifold with good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$. Then the cohomologies of $\mathbf{\Omega}^*(M)$ and $\check{C}^*(\mathfrak{U}, \mathbb{R})$ are isomorphic. That is,*

$$H^*_{dR}(M) \cong \check{H}^*(\mathfrak{U}, \mathbb{R})$$

where $\check{H}^q(\mathfrak{U}, \mathbb{R})$ denotes the q^{th} cohomology of $\check{C}^*(\mathfrak{U}, \mathbb{R})$.

The reason that this is such a profound theorem is because we are relating the geometry of a manifold, with the de Rham complex, to the topology of a manifold, with the Čech complex with values in $\mathbb{R}$. Also, as we have seen above, we are performing calculus in one and purely algebra in the other. In order to attempt this proof we will need one more result.

Theorem 6.2: *Let M be a manifold with cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$. Then the generalised Mayer-Vietoris sequence below is an exact sequence.*

$$0 \to \mathbf{\Omega}^*(M) \xrightarrow{\delta} \check{C}^0(\mathfrak{U}, \mathbf{\Omega}^*) \xrightarrow{\delta} \check{C}^1(\mathfrak{U}, \mathbf{\Omega}^*) \xrightarrow{\delta} \check{C}^2(\mathfrak{U}, \mathbf{\Omega}^*) \to \cdots$$

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Proof. We begin by noting that by the property that $\delta^2 = 0$ we have that

$$\ker(\delta \cap \check{C}^q(\mathfrak{U}, \mathbf{\Omega}^*)) \subset \operatorname{im}(\delta \cap \check{C}^q(\mathfrak{U}, \mathbf{\Omega}^*))$$

To show the converse (and hence equality) is equivalent to showing that given a p-cocycle $\zeta \in \check{C}^p(\mathfrak{U}, \mathbf{\Omega}^*)$ there exists a $\xi \in \check{C}^{p-1}(\mathfrak{U}, \mathbf{\Omega}^*)$ such that $\delta\xi = \zeta$. In order to show this, choose a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ on the cover $\mathfrak{U}$. We define the components of ξ

$$\xi_{\alpha_0\cdots\alpha_{p-1}} = \sum_{\beta\in A} \rho_\beta \zeta_{\beta\alpha_0\cdots\alpha_{p-1}}\big|_{U_{\alpha_0\cdots\alpha_{p-1}}}$$

where we let $\zeta_{\beta\alpha_0\cdots\alpha_{p-1}}$ denote $(-1)^i\zeta_{\alpha_0\cdots\alpha_i\beta\alpha_{i+1}\cdots\alpha_{p-1}}$ for $\alpha_i < \beta < \alpha_{i+1}$ or 0 if $\alpha_i = \beta$ for some i. This should look very reminiscent of the wedge product and follows the same rules – i.e.

$$\zeta_{\alpha\beta} = -\zeta_{\beta\alpha} \tag{1}$$

$$\zeta_{\alpha\alpha} = 0 \tag{2}$$

From above, we apply δ to find

$$\begin{aligned}(\delta\xi)_{\alpha_0\cdots\alpha_p} &= \sum_{i=0}^{p}(-1)^i \cdot \xi_{\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_p}\big|_{U_{\alpha_i}} \\ &= \sum_{i=0}^{p}(-1)^i \sum_{\beta\in A} \rho_\beta \zeta_{\beta\alpha_0\cdots\alpha_i\alpha_{i+1}\cdots\alpha_p}\big|_{U_{\alpha_0\cdots\alpha_p}}\end{aligned}$$

But, because $\delta\zeta = 0$

$$\begin{aligned}(\delta\zeta)_{\beta\alpha_0\cdots\alpha_p} =& \zeta_{\alpha_0\cdots\alpha_p} + \sum_{i=0}^{p}(-1)^{i+1} \cdot \zeta_{\beta\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_p}\big|_{U_{\alpha_i}} = 0 \\ \therefore \zeta_{\alpha_0\cdots\alpha_p} =& \sum_{i=0}^{p}(-1)^i \cdot \zeta_{\beta\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_p}\big|_{U_{\alpha_i}} \\ =& (\delta\xi)_{\alpha_0\cdots\alpha_p}\end{aligned}$$

and hence exactness follows. □

From the result above we get exactness in the rows of the Čech de-Rham double complex. Coupled with the Poincaré Lemma, which gives us exactness along the columns of the double complex when we considering a good cover, we have the correct ingredients to prove de Rham's theorem.

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Proof (de Rham's Theorem). Firstly, let M be a manifold with good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$. We will prove the isomorphism between $H^q_{dR}(M)$ and $\check{H}^q(\mathfrak{U}, \mathbb{R})$ by indirectly proving isomorphisms

1. $H^n_{dR}(M) \cong H^n(K^*)$
2. $\check{H}^n(\mathfrak{U}, \mathbb{R}) \cong H^n(K^*)$

Where $H^n(K^*)$ is the n^{th} cohomology induced on the sequence K^* by the differential operator D from section 4.1— or more explicitly,

$$H^n(K^*) = \ker(D \cap K^n)/\mathrm{im}(D \cap K^n).$$

Let $\zeta = (\zeta_0, ..., \zeta_n) \in K^n = \bigoplus_{p+q=n} \check{C}^p(\mathfrak{U}, \boldsymbol{\Omega^q})$ be a cocycle. It should be noted that with the notation above, $\zeta_i \in \check{C}^i(\mathfrak{U}, \boldsymbol{\Omega^{n-i}})$. We will strive to show that there is some element $\lambda = (\lambda_1, 0, ..., 0) \in K^n$ such that $[\zeta] = [\lambda]$ and then then the isomorphism in (1) will follow. Note that since $D\zeta = 0$ we must have that $\delta\zeta_n = 0$ (directly from the definition of D). Therefore, by δ-exactness there is a $\xi_{n-1} \in \check{C}^{n-1}(\mathfrak{U}, \boldsymbol{\Omega^0})$ such that $\delta\xi_{n-1} = \zeta_n$. Now, consider the cocycle

$$\begin{aligned} \zeta' &= (\zeta_0, ..., \zeta_{n-1} + (-1)^{n+1} \cdot d\xi_{n-1}, 0) \\ &= (\zeta_0, ..., \zeta'_{n-1}, 0) \end{aligned}$$

and notice that

$$\zeta = \zeta' + D(0, ..., 0, \xi_{n-1}) \implies [\zeta] = [\zeta'].$$

We repeat the process on the cocycle ζ'. Since $D\zeta' = 0$ we have that

$$(D\zeta')_n = \delta(\zeta'_{n-1}) + d(0) = \delta(\zeta'_{n-1}) = 0.$$

Again, by δ-exactness, there is a ξ_{n-2} such that $\delta\xi_{n-2} = \zeta'_{n-1}$. We construct the cocycle

$$\begin{aligned} \zeta'' &= (\zeta_0, ..., \zeta_{n-2} + (-1)^{n+1} \cdot d\xi_{n-2}, 0, 0) \\ &= (\zeta_0, ..., \zeta'_{n-2}, 0, 0) \end{aligned}$$

which satisfies

$$\zeta = \zeta'' + D(0, ..., 0, \xi_{n-2}, \xi_{n-1}) \implies [\zeta] = [\zeta''].$$

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After repeating this process n times we will have constructed an element $\zeta^{(n)} = (\zeta_0', 0, ..., 0)$ satisfying

$$\zeta = \zeta^{(n)} + D(\xi_0, ..., \xi_{n-1}) \implies [\zeta] = [\zeta^{(n)}].$$

Since $D\zeta^{(n)} = 0$ we have that $\delta\zeta_0' = 0$ and $d\zeta_0' = 0$. Since $\zeta_0' \in \check{C}^0(\mathfrak{U}, \boldsymbol{\Omega^n})$ and $\delta\zeta_0' = 0$ it must be the case that ζ_0' is a global form ω on M. Also, because δ and d commute, the $d\omega = 0$ and hence $\delta^* : H^n_{dR}(M) \to H^n(K^*)$ is surjective. Injectivity is attained by noting that given a closed global form $\omega \in \boldsymbol{\Omega^n}(M)$, the element $\phi = (\delta\omega, 0, ..., 0) \in K^n$ is a cocycle since d commutes and given $\phi = [\delta\omega', 0, ..., 0]$, $[\phi] = [\phi']$ if and only if $[\omega] = [\omega']$. Hence,

$$H^n_{dR}(M) \cong H^n(K^*)$$

Note that we only used exactness along the rows above. Because the columns are exact, due to the Poincaré Lemma and the fact $\mathfrak{U}$ is a good cover, we can also start with a cocycle $\zeta = (\zeta_0, ..., \zeta_n) \in K^n$ and reduce it to an element $\zeta_{(n)} = (0, ..., 0, \zeta_n')$ satisfying

$$\zeta = \zeta_{(n)} + D(\epsilon_0, ..., \epsilon_{n-1}) \implies [\zeta] = [\zeta_{(n)}].$$

We will now show that the inclusion map $i : \check{C}^n(\mathfrak{U}, \mathbb{R}) \to \check{C}^n(\mathfrak{U}, \boldsymbol{\Omega^0})$, which sends real values to constant functions, such that $i^* : \check{H}^n(\mathfrak{U}, \mathbb{R}) \to H^n(K^*)$ is an isomorphism. That is, given an element φ with the components $\varphi_{\alpha_0 \cdots \alpha_n} \in \mathbb{R}$ we have

$$i\varphi = (0, ..., 0, \varphi)$$

a constant function from $U_{\alpha_0 \cdots \alpha_n} \to \mathbb{R}$. It should be noted that i and δ commute which follows easily from i being an inclusion. Since $D\zeta_{(n)} = 0$, we have that

$$d\zeta_n' = 0, \ \delta\zeta_n' = 0.$$

Since $\zeta_n' \in \check{C}^n(\mathfrak{U}, \boldsymbol{\Omega^0})$ and $d\zeta_n' = 0$, ζ_n' is composed of locally constant functions. Thus, there is an element $\varphi \in \check{C}^n(\mathfrak{U}, \mathbb{R})$ such that $i\varphi = \zeta_{(n)}$. By the commutativity of δ and i, φ must be a cocycle and hence i^* is surjective. Conversely, given a cocycle $\varphi \in \check{C}^n(\mathfrak{U}, \mathbb{R})$, we note that $i\varphi = (0, ..., 0, \varphi)$ is a cocycle under D since

$$(-1)^n \cdot d\varphi = 0, \ \delta\varphi = 0.$$

This gives us that i^* is a bijection and hence,

$$\check{H}^n(\mathfrak{U}, \mathbb{R}) \cong H^n(K^*)$$

completing our proof. □

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7 Appendix

7.1 Proof of the Snake Lemma

Proof. We have already proven the relevant properties of f^* and g^* so it simply remains to show that we can find a function $\partial : H^q(C) \to H^{q+1}(A)$ such that this sequence of cohomology classes is exact.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow d_A & & \downarrow d_B & & \downarrow d_C & & \\
0 & \longrightarrow & A^q & \xrightarrow{f} & B^q & \xrightarrow{g} & C^q & \longrightarrow & 0 \\
 & & \downarrow d_A & & \downarrow d_B & & \downarrow d_C & & \\
0 & \longrightarrow & A^{q+1} & \xrightarrow{f} & B^{q+1} & \xrightarrow{g} & C^{q+1} & \longrightarrow & 0 \\
 & & \downarrow d_A & & \downarrow d_B & & \downarrow d_C & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

We want to send each closed element of C^q to a closed element in A^{q+1} and then show that this map, ∂, is well defined for their associated cohomology classes. So let $c \in C^q$, a closed q-form. By the surjectivity of g there is an element $b \in B^q$ such that $g(b) = c$. Then,

$$
\begin{aligned}
g(d_B b) =& d_C g(b) \\
=& d_C c \\
=& 0
\end{aligned}
$$

so $db \in \ker(g) = \mathrm{im}(f)$. Therefore there is an element $a \in A^{q+1}$ such that $f(a) = d_B b$. Clearly a is closed since $f(d_A a) = d_B f(a) = d_B^2 b = 0$ and by the injectivity of f, $d_A a = 0$. To show this map is well defined, suppose $[c'] = [c]$ for some $c' \in C^q$ or equivalently $c' = c + d\omega$ for some $\omega \in C^{q+1}$. By surjectivity, there is a $b' \in B^q$ such that $g(b') = c + d_C\omega$. In fact, since $g(b' - b) = d_C\omega$, we have that $g(\psi) = \omega$ and hence $g(b') = g(b + d_B\psi) = c'$ for some $\psi \in B^{q-1}$. Note too, that since

$$
\begin{aligned}
g(d_B b') =& d_C g(b') \\
=& d_C c + d_C^2 \omega \\
=& 0
\end{aligned}
$$

and $\text{im}(f) = \ker(g)$, there is an element $a' \in A^q$ such that

$$\begin{aligned} f(a') = d_B b' &= d(b + d_B\psi) \\ &= d_B b + d_B^2\psi \\ &= d_B b \\ &= f(a) \end{aligned}$$

Thus, by the injectivity of f, we have that $a = a'$ and therefore $[a] = [a']$ which implies that $\partial : H^q(C) \to H^{q+1}(A)$ is well defined.

To show exactness we just need to show that $\ker(f^*) = \text{im}(\partial)$ and $\ker(\partial) = \text{im}(g^*)$. To show $ker(f^*) = im(\partial)$, consider $[a] \in \ker(f^*)$ for some $a \in A^{q+1}$. If $f^*([a]) = [0]$ then we must have that $f(a) = d\psi$ for $\psi \in B^q$. Applying g to $f(a)$ we see that

$$g(d_B\psi) = d_C g(\psi) \in C^{q+1} \text{ for } \psi \in C^q.$$

It follows that

$$[a] = \partial([g(\psi)]) \implies \ker(f^*) \subset \text{im}(\partial).$$

Conversely, if we let $[a] \in \text{im}(\partial)$ where $a \in A^{q+1}$, then $[a] = \partial([c])$ for some closed $c \in C^q$. We have shown previously that if $[a] \in \text{im}(\partial)$ then $f(a) = d_B b$. It then follows that

$$f^*([a]) = [f(a)] = [d_B b] \equiv [0] \implies ker(f^*) \supset im(\partial) \implies ker(f^*) = im(\partial)$$

Now, to show $\ker(\partial) = \text{im}(g^*)$, consider $[c] \in \text{im}(g^*)$ where $c \in C^q$. By the construction of ∂, if $\partial([c]) = [0]$ then $0 \in A^{q+1}$ satisfies:

$$f(0) = db \in B^{q+1} \tag{3}$$

$$g(b) = c \in C^q \tag{4}$$

Thus, by (2) we have that $c \in \text{im}(g^*)$, implying that $\ker(\partial) \subset \text{im}(g^*)$. Similarly, if $[c] \in \text{im}(g^*)$ then $d_C g(b)$. Since $f(0) = 0 = d_C g(b)$ where $0 \in A^{q+1}$ and we have that $\partial([g(b)]) = [0]$. So we have that $\ker(\partial) \supset \text{im}(g^*)$ and therefore

$$\ker(\partial) = \text{im}(g^*)$$

Thus the sequence is exact. □

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