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PDE Metric Solution

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Abstract

The following report details the research carried out under the AMSI summer research scholarship in the summer of 2013/2014. A partial differential equation, a generalization of the Sinh-Gordon equation with 3 singularities, was solved numerically on the punctured Riemann sphere using the method of Green's Functions for different values of parameters.

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Introduction

Mathematical physics is a subject that often weaves together diverse areas of mathematics as it attempts to solve physical problems. Recently, a link was found between completely integrable systems and the theory of P.D.E.s (see [1]) which sparked the student's interest in such classes of PDEs and hence served as the impetus for this enquiry. The goal of the research was to consider an important singular PDE and develop methods to solve it numerically. The method is one based on Greens functions and differentiation by parts, involving an iterative formula that would eventually converge to the solution.

In this report, I will first give a full overview of the specific problem considered. I will then present the iterative formulae used, and describe how the problem was solved; briefly touching on the derivation. Then I will move onto the practical considerations of implementing the procedure outlined in the previous section. Finally, there will be a results section, where a typical solution will be presented.

Problem Outline

The equation that we would like to solve is:

$$\frac{1}{4}\Delta f(\boldsymbol{x}) = e^{2f} - \rho^4 B(|\boldsymbol{x}|)e^{-2f} \qquad (1)$$

where:

$$B(|\boldsymbol{x}|) = \frac{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^{2a_3}|\boldsymbol{x}_2 - \boldsymbol{x}_3|^{2a_1}|\boldsymbol{x}_1 - \boldsymbol{x}_3|^{2a_2}}{|\boldsymbol{x} - \boldsymbol{x}_1|^{2-2a_1}|\boldsymbol{x} - \boldsymbol{x}_2|^{2-2a_2}|\boldsymbol{x} - \boldsymbol{x}_3|^{2-2a_3}}$$

We work in the real plane with 3 punctures at the singular points. The $\boldsymbol{x}_i$ ($i = 1,2,3$) are specific points (the locations of the singularities) and the a_i are parameters satisfying the conditions:

$$a_1 + a_2 + a_3 = 2$$
$$0 < a_i < 2$$

ρ is a positive parameter that can in theory vary from zero to infinity, but in practice $\rho = 3/10$ is already considered a large value.

There is an interesting interpretation of the solution function to the system (1). The solution, f, can be regarded as a metric if viewed as a function on the Riemann sphere that has been stereographically projected onto the complex plane (which is equivalent to the real plane we are working on). To illustrate this, a solution to

$$\frac{1}{4}\Delta f(\boldsymbol{x}) = -e^{2f}$$

with appropriate conditions at infinity will be the usual metric $Sin(\theta)d\theta d\varphi$ with constant positive curvature on the sphere, whereas a solution to

$$\frac{1}{4}\Delta f(\boldsymbol{x}) = e^{2f}$$

again with certain asymptotic conditions will be a metric with constant negative curvature on the sphere. Our function satisfies the asymptotic conditions:

$$f(\boldsymbol{x}) \approx -2Log(|\boldsymbol{x}|) + f_\infty \quad |\boldsymbol{x}| \to \infty \qquad (2.1)$$

$$f(\boldsymbol{x}) \approx 2m_i Log(|\boldsymbol{x} - \boldsymbol{x}_i| + f_i \quad |\boldsymbol{x} - \boldsymbol{x}_i| \to 0 \qquad (2.2)$$

$$m_i \equiv p_i - \frac{1}{2}$$

and has a rather more complicated interpretation, but still, that of a metric.

There are certain natural questions that arise when one is confronted with such a system. Is the problem well posed? In other words does there exist a unique solution and to what class of functions does it belong? The answer to these questions can intuitively be partially found by means of a detour into the theory of electrostatics.

In 2 dimensions, an analogy to the well-known Coulomb's Law states that the electrostatic attraction between two point charges will be proportional to the inverse of the distance between them, or

$$\boldsymbol{E} \sim \frac{1}{r}\hat{\boldsymbol{r}}$$

The potential difference of two point charges will thus be a logarithmic function

$$V \sim Log(|r|)$$

We will also need the differential equation that the potential satisfies

$$\Delta V(\boldsymbol{x}) = \frac{1}{\epsilon_0}\rho(\boldsymbol{x})$$

Here, $\rho(\boldsymbol{x})$ is the charge density.

Now let's return to our original problem. The equation (1) along with the asymptotic conditions (2) can be interpreted as the electrostatic problem of finding the potential energy of a charge distribution $e^{2f} - \rho^4 B(|\boldsymbol{x}|)e^{-2f}$ and 3 point charges placed at $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ along with a specification of the field at infinity. If the right hand side of (1) was not dependent on the function f itself, this would obviously have a unique, continuous solution, since such an arrangement can be constructed in the laboratory (albeit a 2 dimensional one). The only difficulties that arise is that the charge distribution depends on the field itself, f, and the rational function $B(|\boldsymbol{x}|)$ has singularities. We note that, by substituting the appropriate asymptotic expressions into the right hand side of equation (1), it can be seen that $B(|\boldsymbol{x}|)$ in fact weakens the singularity of the right hand side. Therefore, it is unlikely that this term would make the problem malicious. So the electrostatic interpretation goes far into addressing the issue, though not completely. However, there are other physical interpretations that ensure the well – posed nature of our problem.

Solution Method

As mentioned in the previous section, the problem under consideration is very similar to the electrostatic problem of finding the potential from a given charge distribution. Therefore, in deriving the solution, mathematical methods are used that are common in electrostatics, including various Green's theorems (whose analogue in electrostatics is Gauss' Law) and the method of Greens functions. In particular, we use an iterative approach that was derived using such mathematical methods. The appendix of any book on partial differential equations such as [2] can be consulted to review the basic multidimensional calculus facts (in particular the Green's theorems) used extensively in this derivation.

We give a sample of how the main recurrence was derived, so that the reader may get a feel for what is involved.

Let us first introduce the Green's function for the operator $\frac{1}{4}\Delta$

$$G(\boldsymbol{x}-\boldsymbol{x}') = \frac{2}{\pi} Log(|\boldsymbol{x}-\boldsymbol{x}'|)$$

which satisfies

$$\frac{1}{4}\Delta G(\boldsymbol{x}-\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}')$$

and consider the integral expression, where the domain of integration is the whole real plane

$$\frac{1}{4}\int G(\boldsymbol{x}-\boldsymbol{x}')\Delta f dV = \int G(\boldsymbol{x}-\boldsymbol{x}')(e^{2f} - \rho^4 B(|\boldsymbol{x}'|)e^{-2f})dV'$$

Using a Green's theorem, we can transfer the Laplacian from f to the Green's function. This will yield some boundary terms but, most importantly, will result in the convolution of f with the function $\delta(\boldsymbol{x})$ which simplifies to $f(\boldsymbol{x})$. Rearranging to make f the subject, and using the asymptotic boundary conditions for f to evaluate the boundary terms, being very careful to note the contributions from the singular points $\boldsymbol{x}_i$, yields:

$$f(\boldsymbol{x}) = \int G(\boldsymbol{x}-\boldsymbol{x}')(e^{2f(x')} - \rho^4 B|\boldsymbol{x}'|e^{-2f(x')})dV' + f_\infty + \sum_{i=1}^{3} 2m_i Log|\boldsymbol{x}-\boldsymbol{x}_i| \quad (3)$$

So the solution function is a fixed point of the transformation given by (3). If we assume that in some domain, (3) is a contraction mapping and we define

$$f_n = \int G(\boldsymbol{x}-\boldsymbol{x}')(e^{2f_{n-1}} - \rho^4 B|\boldsymbol{x}'|e^{-2f_{n-1}})dV' + f_{n-1,\infty} + \sum_{i=1}^{3} 2m_i Log|\boldsymbol{x}-\boldsymbol{x}_i| \quad (4)$$

With an appropriate choice of f_0, we will have that $f_n \to f$ as $n \to \infty$ where f is the solution to (1). To avoid confusion, we state explicitly that in the above expression $f_{n-1,\infty}$ denotes a constant and f_{n-1} a function.

There are other iteration formulas for the constants f_∞, f_i $(i = 1,2,3)$ and we omit a discussion of these here since they are derived in a similar fashion. A more complete step-by-step overview of the method used is included in the appendix.

In summary, to solve the system (1), we start with a "guess" solution, f_0, and use the recurrence relation (3) to find f_n which will either converge to our solution or diverge depending on our choice of f_0.

Numerical Implementation

To carry out the above transformations, we need to define an appropriate (bounded) domain on which to act out transformations. We choose a ball of radius R with small punctures at the singularities

$$D = B_R(0)/\{B_\varepsilon(\boldsymbol{x}_1) \cup B_\varepsilon(\boldsymbol{x}_2) \cup B_\varepsilon(\boldsymbol{x}_3)\}$$

where $\varepsilon > 0$ is small. Since the domain of integration in (4) is the whole real plane, we numerically integrate over D and analytically add the extra contributions from D^c using the asymptotic expansions in (2). The mesh we use is an adaptive triangulation. Since the function has singularities near $\boldsymbol{x}_i$, after a set number of iterations we refine the mesh where the function is large (and hence where the errors are likely to be most severe). Typically, this means that after a few such refinements the mesh is dense near the singularities $\boldsymbol{x}_i$. The image below shows the triangulated domain, and clearly displays the increase in mesh density close to the singularities. ε, has been chosen to be large for the clarity of the image, and it in no way reflects actual values of ε used in the calculations.

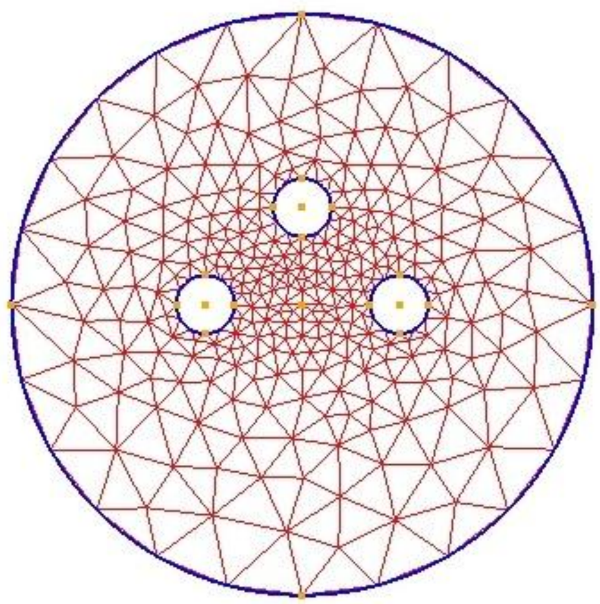


Fig 1. An example of the domain over which the numerical calculations were performed as well as a typical mesh of the domain.

A free open source meshing software, GMSH, was used which can be found by the link given in [3], and the programing platform was Fortran. Additionally, parallel programming techniques were implemented to ensure our developed software could be run quickly and efficiently on multiple cores. The calculations were done on the NCI supercomputer located at the Australian National University.

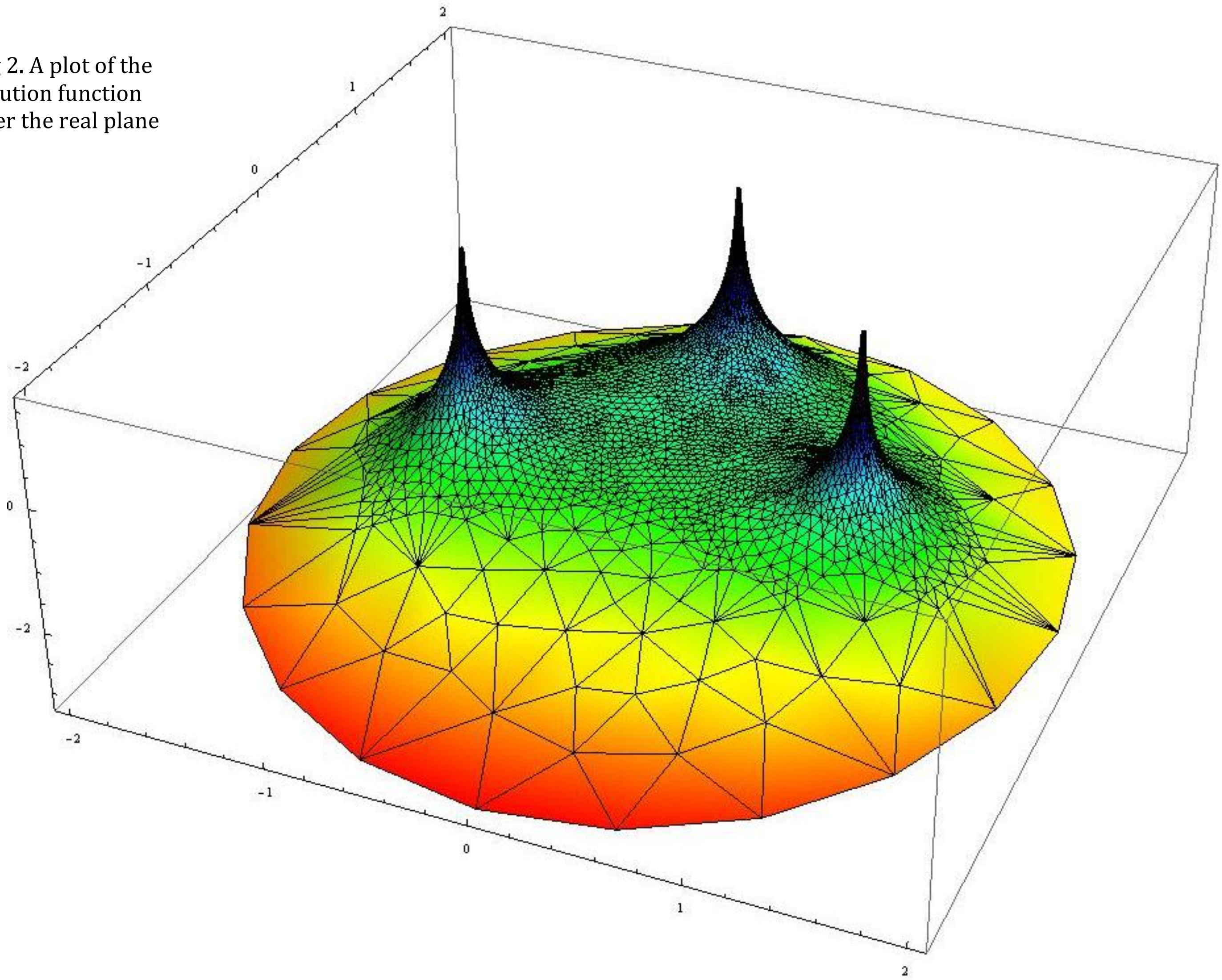
There are some technical issues regarding the implementation of the algorithm described by equation (4). Below, once such issue is detailed so that the reader may get an idea of the typical problems encountered in numerical methods.

The choice of initial function, f_0, greatly determines the convergence of the system and as the parameter ρ increases, a simple guess of f_0 is inadequate. The reason behind this is due to the singularities inherent in the rational function $B|\boldsymbol{x}'|$. Although we mentioned previously that this term in fact tempers the singularities at $\boldsymbol{x}_i$, if f_0 is defined as an arbitrary initial function, say a constant function, the solution quickly diverges for relatively small values of ρ. An explanation of this is that the function f_0 has no singularity itself which, when coupled with $B|\boldsymbol{x}'|$, creates a weak singularity. This problem can be solved by ensuring the initial function f_0 already has singularities at $\boldsymbol{x}_i$, and practically this means solving the system (1) for $\rho = 0$ and then using this solution as the initial function for the subsequent calculations when ρ is increased.

Results

The equation (1) was solved for various values of ρ and fixed values of $a_i = \frac{2}{3}$ and $p_i = \frac{1}{12}$ $\quad (i = 1,2,3)$. A plot of the solution is given on the next page. Note the logarithmic singularities where the mesh density is high and the function approaches infinity. Convergence was displayed, and certain analytic checks confirmed that this in fact was the sought after solution.

Fig 2. A plot of the solution function over the real plane



Conclusion

Over the summer of 2013/2014, a method was developed for the solving of a singular partial differential equation. The method was successful and a numerical result was obtained. Further research would involve solving such equations with more singularities and solving more general equations, as well as optimizing the numerical calculations.

I thank AMSI for providing me with the scholarship and the opportunity to present my findings in the Big Day In. I also thank my supervisor Vladimir Bazhanov, who was always approachable, Vladimir Mangazeev, who helped me reach the level I am at now and Andrey Bliznyuk for his kind assistance with the programming implementation.

Appendix

Below we give a complete list of the formulae and a breakdown of the recurrence procedure used.

1. Define initial guess function f_o and initial guess constants f_∞, f_1, f_2, f_3.

2. Compute, with $n = 1$,

$$\varphi_n = \int G(\boldsymbol{x} - \boldsymbol{x}')(e^{2f_{n-1}} - \rho^4 B|\boldsymbol{x}'|e^{-2f_{n-1}})dV' + f_{n-1,\infty} + \sum_{i=1}^{3} 2m_i Log|\boldsymbol{x} - \boldsymbol{x}_i|$$

3. Compute for $f_{n,\infty}$, (with $n = 1$)

$$f_{n,\infty} = \frac{1}{2} Log(\frac{E + \sqrt{E^2 + 4DC}}{2C})$$
$$C = \int e^{2\varphi} dV$$
$$D = \rho^4 \int e^{-2\varphi} dV$$
$$E = \pi(-(m_1 + m_2 + m_3) - 1)$$

   Note that all quantities, E, C, D, are positive so $f_{n,\infty}$ is well defined.

4. Compute for $f_{n,i}$ (with $n = 1, i = 1,2,3$).

$$f_{n,i} = \mathrm{G}_i(\varphi, f_{n-1,1}, f_{n-1,2}, f_{n-1,3})$$

   where G_i is a rather complicated function that we would prefer to omit.

5. Compute for f_n $(n = 1)$

$$f_n = \varphi_n + f_{n,\infty}$$

6. Repeat steps 2-5 until f_n becomes a fixed point.

References

[1] Bazhanov, V & Lukyanov, S 2013, '*Integrable structure of Quantum Field Theory: Classical flat connections versus quantum stationary states*', [arXiv:hep-th/1310.4390].

[2] Lawrence L, 2010, *'Partial Differential Equations',* second edition, American Mathematical Society.

[3] Geuzaine, C & Remacle, J, *'Gmsh: a three-dimensional finite element mesh generator with built in pre- and post-processing facilities'.* Available from: <geuz.org/gmsh/>. [17/12/2013]