The fundamental groupoid

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Abstract

The fundamental groupoid can be viewed as a generalization of the fundamental group. In this report, we take a different approach by considering the fundamental groupoid as a category. After introducing the basic concept of category theory, we will construct the fundamental groupoid and generalize a more general version of Van Kampen's theorem. The final result is that we can calculate the fundamental group of a circle using the improved Van Kampen's theorem.

1 Preliminaries

Definiton 1.1. A category C consists of :

1. a class $Ob(C)$, called the objects of C,
2. for each $x, y \in Ob(C)$, a set $C(x, y)$ is called the set of morphism in C from x to y,
3. a binary operation, called composition. For any $a, b, c \in Ob(C)$, if $f \in C(a, b), g \in C(b, c)$, then the composition $gf \in C(a, c)$.

The composition is associative and the identity morphism 1_x exists for every object x in C.

Similar to other mathematical object, there is a notion of a category contained in a bigger category. Let C, D be categories. We say D is a subcategory of C if

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1. each object of D is an object of C,
2. for each $x, y \in Ob(D)$, $D(x, y) \subseteq C(x, y)$.
3. the composition of morphism D is the same as for C, and
4. the identity morphism behaves the same.

A subcategory D is called $full$ if $D(x, y) = C(x, y)$ for all $x, y \in Ob(D)$.

Examples.

- Set as category with sets as objects and functions as morphisms.
- Top with sets as object and continuous maps as morphisms.
- Grp with groups as object and group homomorphisms as morphisms.
- $Grpd$ with groupoids as object and functors between groupoid as morphisms.

Not that we have a class as objects in these examples but we can also have a set as object. For example : a group is a category with only one object and all the morphisms are invertible. And a groupoid is exactly the same except it could have more than one object.

The last example tells us more about category. Since a groupoid is a category, it shows us that we can construct a category using categories as object. The functors is like a map between categories.

Definiton 1.2. Let C, D be categories. A functor $f : C \to D$ maps each object $x \in C$ to an object fx in D. It also maps each morphism $a \in C(x, y)$ to a morphism $fa \in D(fx, fy)$. We then call fa the morphism induced by a. The functor F must preserve:

- Composition. If $f : x \to y$ and $g : y \to z$, then $F(gf) = F(g)F(f)$.
- Identity morphism. For all $x \in Ob(C)$, $F(1_x) = 1_{Fx}$.

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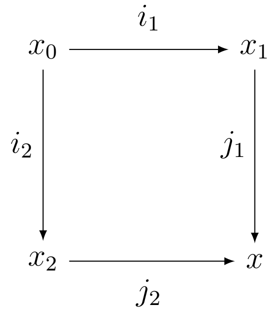
From the definition, we can show that any functor $f : C \to D$ preserve isomorphism. If a morphism a is an inverse of b in category C (i.e: $ab = 1$, $ba = 1$), then by the above definition $fafb = 1$ and $fbfa = 1$. This shows that a topological invariant is no more than a functor from Top to other category. The fundamental groupoid is then a functor $\pi : Top \to Grpd$.

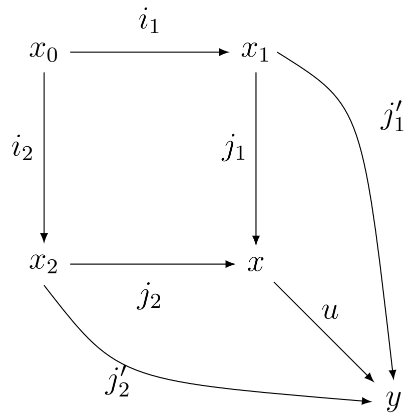
The last thing we need to know about category theory is a special commutative diagram, called pushout.

Definiton 1.3. Let C be a category. A diagram



where x_0, x_1, x_2, x are objects and i_1, i_2, j_1, j_2 are morphism, is a pushout if

1. it commutates.
2. for every object $y \in C$ and a pair of morphism $j_1' : x_1 \to y$ and $j_2' : x_2 \to y$ such that the following outer diagram commutes,



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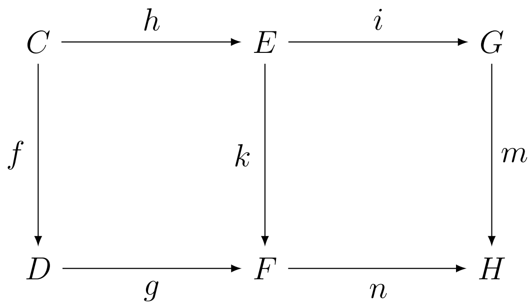
that is $j_1' i_1 = j_2' i_2$. Then there exist a unique morphism $u : X \to Y$ such that the whole diagram.

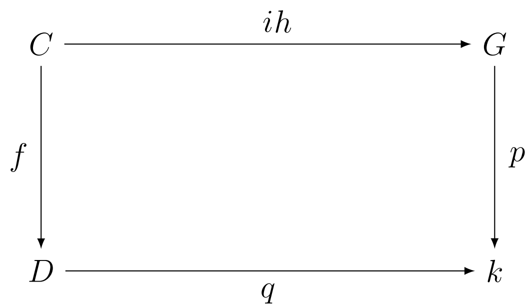
The next proposition roughly states that a composition of pushout is a pushout.

Proposition 1.4. Suppose we are given a commutative diagram in **C** where the first and second square is a pushout.



Then the outer square is a pushout.

Proof. Suppose we are given an arbitrary object K and morphisms $p : G \to K$, $q : D \to K$ such that the following diagram commutates.



Since it satisfies the pushout properties of the first square, there is a unique morphism $a : F \to K$ such that $ag = q, ak = pi$. By the pushout property of second square, there is a unique morphism $b : H \to K$ such that $p = bm, a = bn$.

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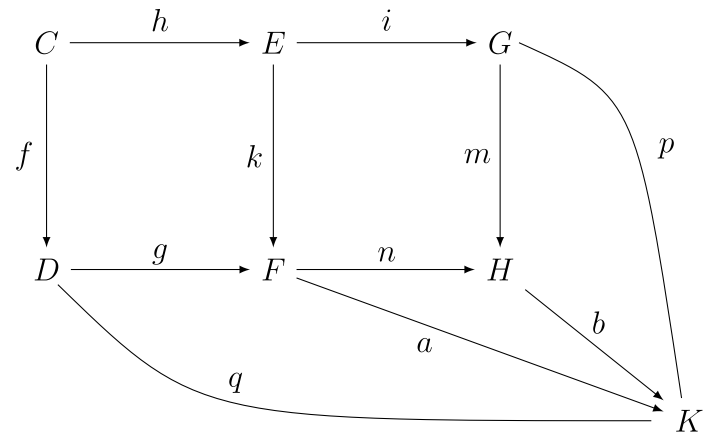

It follows that $b = pm$, $q = bng$, thus the morphism $b : H \to K$ is the required morphism for the outer square.

(Uniqueness) Suppose $b' : H \to K$ is another morphism such that $b'm = p$, $b'ng = q'$. Then $b'n$ is another morphism from F to K and by the uniqueness of morphism a, $a = bn$. Similarly, $b' = b$ by the uniqueness of morphism b.

□

To avoid confusion, we used categories as object and functors as morphisms in the above theorem.

2 Fundamental Groupoid

The fundamental groupoid is a close relative of the fundamental group. While in the latter one studies the loops in a topological space, the fundamental groupoid focuses on the paths. More precisely, it is defined to be the collection of equivalence class of paths in a space. However, we first need an equivalence relation to classify the paths.

2.1 Homotopy of paths

For a path a from x to x' in topological space X, it is defined to be a map $f : [0, r] \to X$ such that $f(0) = x$ and $f(r) = x'$. We say that the real number r is the length of a path.

Definiton 2.1. Let a,b be paths in X of length r with same endpoints. A homotopy of length q is defined to be a map $F : [0, r] \times [0, q] \to X$ such that for $s \in [0, r]$.

$$F(s, 0) = a(s) \qquad\qquad F(s, 1) = b(s).$$

The above definitions might look different to the one using interval $[0, 1]$ but the more general form turns out to be convenient in defining the composition of paths. Furthermore, there is a continuous surjection from $[0, 1] \to [0, q]$. So it is sufficient to only consider the homotopies of length 1. Now it can be shown that:

- Homotopy of length 1 is an equivalence relation of paths with the same length.
- Homotopy behaves well with respect to composition.

How about the paths with different lengths but same endpoints? To extend our classification, we need to know how two paths can add or equivalently, be composed.

Suppose a,b are two paths with length p, q respectively, the addition of path is defined by the map $a + b : [0, p + q] \to X$, such that

$$(a + b)(s) = \begin{cases} a(s) & \text{if } 0 \leq s \leq p \\ b(s) & \text{if } p < s \leq p + q \end{cases}$$

Now we can complete the classification of paths.

Definiton 2.2. Given two paths a, b that have different lengths. We say a, b are equivalent if there exist constant paths r, s, such that $r + a$, $s + b$ have the same length and are homotopic.

This definition gives an equivalence relation on the path in X.

2.2 Category $PX, \pi X$

We start our construction by defining a new category PX. It has the element of X as object and the paths in X as morphism. Composition of paths is the path addition that we defined above. We can easily verify that the identity and associative axioms hold.

The fundamental groupoid is obtained by applying the equivalence relation 2.2 on the morphisms of category PX.

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Theorem 2.3. The category πX whose objects are the set of points X and whose morphisms are path classes in X is a groupoid. This groupoid is called the fundamental groupoid of X.

At first glance, the fundamental groupoid is much bigger than the fundamental group. But as a trade off, it is harder to compute. In the case of circle $\mathbb{S}^1$, the path classes starting and ending at each pairs of points are isomorphic to $\mathbb{Z}$, and there are infinitely many pairs of points.

To avoid the lengthy computation, we will investigate the subcategory of fundamental groupoid. But we should first give some example of fundamental groupoid of simple spaces.

Examples.

- If the path-components of a space X consists only one point, then $\pi X(x, y) = \phi$ if $x \neq y$. Otherwise, it has constant path. Such a groupoid is called a *discrete* groupoid.
- If the space X is convex, then all paths with the same end points are equivalent. Thus $\pi X(x, y)$ has exactly one element for all x, y in X. A groupoid with this property is called 1-connected or a tree groupoid.

Among those examples, the 1-connected groupoid with two objects $0, 1$ is important. It will be used to describe a more general definition of homotopy (functor). And is an important tool of our final example. We shall denote this groupoid as I and its unique morphism from 0 to 1 by $\imath$.

2.3 The Fundamental subgroupoid πXA

Given a subset A of X, the fundamental subgroupoid πXA is defined to be the full subcategory of πX whose objects are the set A. Equivalently, we can think about it as considering *only* the path classes in X whose has endpoints are in A.

The following figure in which the shaded region represents A, illustrated the idea:

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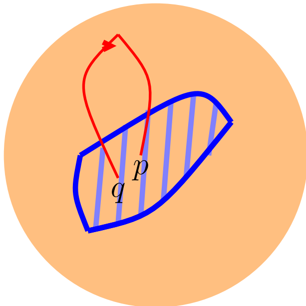

Note that if the subset A of X consists of a single point p, then all the paths in $\pi X\{p\}$ become loops. When we apply the equivalence relation on those loops, we obtain the fundamental group $\pi(X, p)$. Thus, the fundamental subgroupoid $\pi X\{p\}$ is the fundamental group $\pi(X, p)$.

3 Homotopy

As mentioned previously, the fundamental groupoid is hard to calculate, and an appropriate $'size'$ of subset A has yet to be determined. In this section, we introduce the homotopy of functors which enable us to define an equivalent but simpler fundamental subgroupoid. Then we can define what it means for a subgroupoid to be a 'deformation retract' and determine the appropriate properties for the subset A.

Definiton 3.1. Let C and D be categories. A homotopy of functors from C to D is a product functor $F : C \times I \to D$. The initial functor of F is $f = F(\ , 0)$ and the final functor is $g = F(\ , 1)$, where I is the 1-connected groupoid with only two objects. We then write $F : f \simeq g$ and say that f and g and homotopic.

Note: We write an object of D when both arguments of F are objects and a morphism of D when both arguments of F are morphisms.

The groupoid I has a unique morphism $\imath : 0 \to 1$. This allows us to complete characterize the homotopy F with initial and final functors f, g and an invertible morphism θx of F. Here $\theta x = F(x, \imath)$ where x is the identity morphism 1_x in C.

Take a morphism $a : x \to y$ in C, then we have $ga(gx) = gy$. However, we can show that:

$$(ga) \quad = \quad (\theta y)(fa)(\theta x^{-1}). \tag{1}$$

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Proof.

$$\begin{aligned}(\theta y)(fa)(\theta x^{-1})(gx) &= F(y,\imath)F(a,0)F(x,\imath^{-1})F(x,1)\\ &= F(yax(x),\imath 0\imath^{-1}(1))\\ &= F(y,1)\\ &= gy.\end{aligned}$$

□

With the invertibility of θx, we showed that g is determined by and f and θx. Thus for any homotopy of functors F and initial functor f, the functor g is defined by (1). We call θ a homotopy function from f to g, and write $\theta : f \simeq g$.

As a categorical analogue of homotopy of spaces, we inherit most of the latter's properties. They can be demonstrated using the homotopy function θ that we just introduced.

- Homotopy of functors is an equivalence relation on functors $C \to D$.
- Let $f : C \to D, g : D \to \varepsilon, h : \varepsilon \to F$ be functors and suppose $g \simeq g'$. Then $hgf \simeq hg'f$.

On the other hand, the notation of homotopy inverse and homotopy equivalent play a bigger role here. Let C, D be categories and $f : C \to D$, $g : D \to C$ be functors. If $fg \simeq 1_C$ and $gf \simeq 1_D$, then we said f is a homotopy inverse of g. When an inverse exists, we call f a homotopy equivalence, and the categories C, D are said to be homotopy equivalent. We denote this by $C \simeq D$.

Similarly, we can ask for an addition property of homotopy of functors that is relative to some subcategory D. On the subcategory D, the homotopy F will be constant or equivalently the homotopy function θx is an identity morphism. Thus, the initial and final functor agree on D.

Finally, we have a definition of deformation retract and a theorem about it. We can replace a category with a simpler but homotopically equivalent one. This is frequently useful in calculating the fundamental groupoid as πX is complicated after all.

Definiton 3.2. Given that D is a subcategory of C, a functor $r : C \to D$ is called a deformation retraction if $ir \simeq 1_C \ relD$, where $i : D \to C$ is the inclusion functor.

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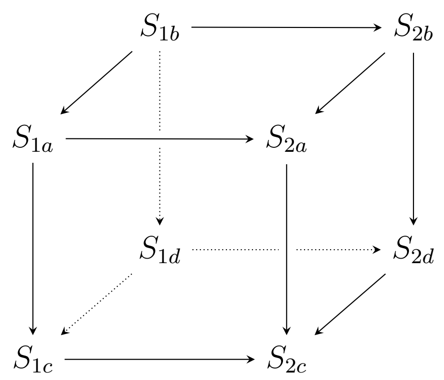
Note that a deformation retraction is automatically a retraction since $ri = r1_C|D = 1_D$, and the categories C and D are homotopy equivalent. But to describe a deformation retract, we want each object of C to be isomorphic to an object of its subcategory D. We then say D is representative in C.

Theorem 3.3. A subcategory D of C is a deformation retract of C if and only if D is a full representative subcategory.

This theorem shows us the appropriate size of the subset A. As πXA itself is a full subcategory, all we need is A to be representative in X. It also turns out that a deformation retract of a pushout is also a pushout. However, retraction does not make sense in the category of commutative squares so we need a further definition.

3.1 Category $C_\square$

Given a category C, we define a category of commutative squares $C_\square$, in which the objects are commutative squares of C. Let S_1,S_2 be two commutative squares. A morphism $s : S_1 \to S_2$ is a collection of maps in C which preserves the square's structure and such that the following cubical diagram commutes.



It can be verified that this is a category. Now we have the notion of retraction of commutative squares and we can state the next theorem.

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Theorem 3.4. Let S_1, S_2 be commutative squares in C such that S_2 is a pushout. If there is a retraction $S_2 \to S_1$, then S is a push out.

The proof will be refer to Brown's book, page 238.

4 Van Kampen's theorem

Van Kampen's theorem is a crucial tool to calculate complicated fundamental groups. It states that the fundamental group $\pi(X, p)$ can be expressed in terms of the fundamental group of its open cover if certain properties hold. Our aim is to state and prove a more general version of this theorem, using the techniques that we developed so far. We will use this theorem to compute the fundamental group of a circle which the original Van Kampen's theorem does not allow.

Theorem 4.1. Let X_0, X_1, X_2 be subspaces of a topological space X such that $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. If A is representative in X_0, X_1, X_2, then the following diagram is a push out.

$$\begin{array}{ccc} \pi X_0 A & \xrightarrow{\;i_1\;} & \pi X_1 A \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle u_1} \\ \pi X_2 A & \xrightarrow{\;u_2\;} & \pi X A \end{array}$$

As the fundamental group is a special case of the fundamental groupoid, it is not surprising that Van Kampen's theorem can be stated in terms of subgroupoid and pushout. However, there is some essential difference. Even though it is usual to assume X to be path-connected, we don't require its subspaces to be path-connected. Instead, we need the set A to be representative in each of its subspaces. We also assume A is a subspace of X as any point outside X makes no difference to πXA.

The proof of 4.1 is constructed in two parts. We first prove the case $A = X$, then the general case using theorem 3.4. The first part of the proof is lengthy and totally topological. For the reader who is interested in it, please see Brown's book, page 242.

Proof. The general case - We start by proving the commutative square πXA is a retraction of πX then apply theorem 3.4. First note that each $\pi X_i A$ is a full subcategory of πX_i $(i = 0, 1, 2,)$. This leaves us to prove that they are representative.

Since A is representative in each X_i, we can always choose a path class from some point in A to each point in X_i. Because every path class has a inverse (i.e: we have a groupoid), each object of πX is isomorphic to an object in πX. By 3.3, each $\pi X_i A$ is a deformation retract and thus also a retract. $\square$

Suppose now that we can restrict A to a single point, then the last theorem is exactly the original Van Kampen's theorem. But to compute the fundamental group of a circle, we need another pushout.

Theorem 4.2. Let C,D be categories and $f : C \to D$ be a functor such that it is injective in objects. Then for any full representative subcategory C' of C, we obtain a push out.

$$\begin{array}{ccc} C & \xrightarrow{\quad r \quad} & C' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ D & \xrightarrow{\quad r' \quad} & D' \end{array}$$

f' is the restriction of f and r, r' are deformation retraction.

We will first prove the above square is commutative. The proof of pushout will be will be referred to brown's book.

Proof. We first construct D' as a full subcategory of D with:

$$Ob(D') \quad = \quad f(Ob(C')) \cup (Ob(D) \setminus f(Ob(C)).$$

For every object $d \in Ob(D) \setminus Ob(D')$, $d = f(c)$ for some $c \in Ob(C)$. Since f is injective and C' is representative in C, the object d is isomorphic to some object $f(ob(c'))$ in D. Therefore, D' is representative in D and by theorem 3.3, there exists a deformation retract $r' : D \to D'$.

Thus we have the following commutative diagram:

$$\begin{array}{ccc} C' & \xrightarrow{\quad i \quad} & C \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ D' & \xrightarrow{\quad j \quad} & D \end{array}$$

where i, j are inclusion. The morphism $r : C \to C'$ is a deformation retract and $\theta : ir \simeq 1 \text{ rel } C'$ is the induced homotopy function.

Similarly, we try to construct a homotopy function $\varphi : jr' \simeq 1 \text{ rel } D'$. We set $\varphi y = f\theta x$ if $y = fx$ and otherwise $\varphi y = 1_y$. Thus φy is invertible and $\varphi f = f\theta$.

Take a morphism $a \in C(x, x')$, observe that:

$$\begin{aligned} jf'r(a) &= fir(a) \\ &= f((\theta x')^{-1} a(\theta x)) \\ &= (f\theta x')^{-1} fa(f\theta x) \\ &= (\varphi fx')^{-1} fa(\varphi fx) \\ &= jr'f(a). \end{aligned}$$

Note that we are actually working on subcategory so we can drop the inclusion j from both sides. Therefore $f'r = r'f$ and the square commutes.

□

The next theorem sums up what we developed so far and allows us to calculate the fundamental group of a circle.

Theorem 4.3. Let A_1 be a subset of A that is representative in X_1. Then we have a pushout diagram.

$$\begin{array}{ccccc} \pi X_0 A & \xrightarrow{\quad i_1 \quad} & \pi X_1 A & \xrightarrow{\quad r \quad} & \pi X_1 A_1 \\ {\scriptstyle i_2}\big\downarrow & & \big\downarrow{\scriptstyle u_1} & & \big\downarrow{\scriptstyle u_1'} \\ \pi X_2 A & \xrightarrow[\quad u_2 \quad]{} & \pi X A & \xrightarrow[\quad r' \quad]{} & \pi X A_1 \end{array}$$

Note: As discussed before, we assume A to be a subset of X.

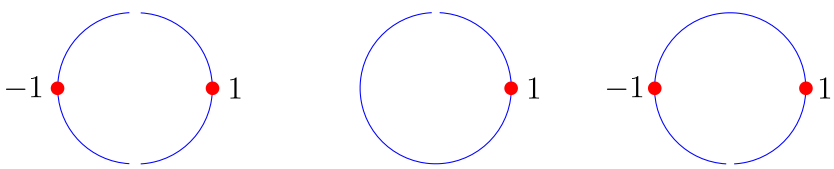
Proof. The proof is almost immediate. By theorem 4.1 and 4.2, the left square and right square are pushout respectively. Since a composition of pushout is again a pushout, the outer square is a pushout.

□

Without the last theorem, the calculation of the fundamental group of a circle is lengthy and complicated. The final example simplifies the lengthy proof to a simple application of 4.3.

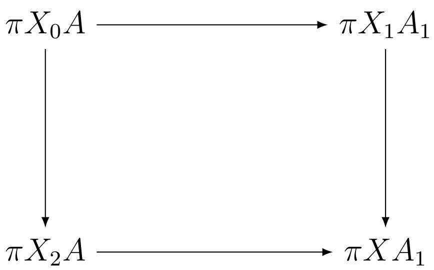
Example. The fundamental group of a circle is isomorphic to $\mathbb{Z}$.

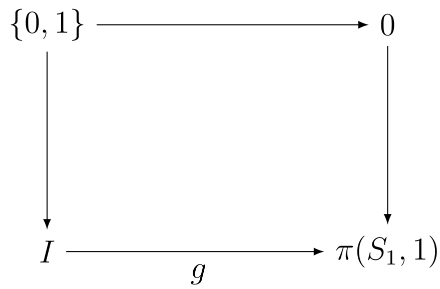
Consider a circle S^1 in $\mathbb{C}$. Let $X_1 = \mathbb{S}^1 \setminus \{i\}$, $X_2 = \mathbb{S}^1 \setminus \{-i\}$ and $X_0 = X_1 \cap X_2 = \mathbb{S}^1 \setminus \{i, -i\}$. We choose the subset $A = \{-1, 1\}$ such that it is representative in X_i $(i = 0, 1, 2,)$. $A_1 = \{1\}$ which is representative in X_1.



X_0A $\qquad$ X_0A_1 $\qquad$ X_2A

By 4.3, the left square is a push out but we can simplify it.





Observe that $\pi X_1 A_1$ is the fundamental group of a simply-connected space, thus it is isomorphic to the trivial group(oid) 0. $\pi X_0 A$ has a point on both simply-connected path-components. Therefore it is isomorphic to discrete groupoid $\{0, 1\}$.
$\pi X_2 A$ has two objects {i},{-i} and has only one path class between them. It is then isomorphic to groupoid I.

Consider the right pushout and a group $\mathbb{Z}$. If there is a morphism $f : I \to \mathbb{Z}$, then there is a unique morphism $h : \pi(S_1, 1) \to \mathbb{Z}$ such that $hg = f$. It was shown in Brown's paper that h is indeed an isomorphism. Therefore, $h : \pi(S_1, 1) \simeq \mathbb{Z}$

5 Conclusion

The aim of this report is to give an introduction to fundamental groupoid. We first study category theory for necessary knowledge to construct the fundamental groupoid. However, it is too hard to calculate so we replace it with a simpler but equivalent sub-groupoid. At last, the improved Van Kampen's theorem tells us that the fundamental groupoid more versatile than the fundamental group.

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