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An investigation into the real 3 dimensional Heisenberg group

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1 The Heisenberg group

The real three dimensional Heisenberg group is a group in the classical sense. It is a space of 3×3 matrices defined in the following way:

$$\mathcal{H}^3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

If we take the variables x, y, z to represent the standard basis directions in the Euclidean space, we can then study the geometry of the group. For notational convenience, the matrices will be denoted in three tuple form for the rest of the report. The group operation is the standard matrix multiplication, which gives the multiplication rule:

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

The identity element of the group is the standard identity matrix $I = (0, 0, 0)$, which corresponds to the origin, and for any matrix (x, y, z), the inverse is given by $(x, y, z)^{-1} = (-x, -y, xy - z)$. The Heisenberg group is a Lie group, as such it has an associated Lie algebra called the Heisenberg algebra.

2 The Heisenberg algebra

The Heisenberg algebra, $\mathfrak{h} := (H, [\cdot,\cdot])$ is a space of matrices equipped with the Lie bracket operation defined in the following way:

$$H := \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\},\ [\cdot,\cdot] : H \times H \to H : [A, B] = AB - BA$$

Clearly $\mathfrak{h}$ is a linear space. To prove that $\mathfrak{h}$ is a Lie algebra, we show that the bracket $[\cdot,\cdot]$ is bilinear, skew-symmetric and satisfies the Jacobi identity.

First we check bilinearity.
Let $A, B, C \in H$ and let $s, t \in \mathbb{R}$, then

$$\begin{aligned} [sA + tB, C] &= (sA + tB)C - C(sA + tB) \\ &= sAC + tBC - sCA - tCB \\ &= s(AC - CA) + t(BC - CB) \\ &= s[A, C] + t[B, C] \end{aligned}$$

Also,

$$\begin{aligned} [C, sA + tB] &= C(sA + tB) - (sA + tB)C \\ &= sCA + tCB - sAC - tBC \\ &= s(CA - AC) + t(CB - BC) \\ &= s[C, A] + t[C, B] \end{aligned}$$

So the Lie bracket is $\mathbb{R}$ bilinear.
Next we check skew-symmetry.

Let $A, B \in H$, then

$$\begin{aligned} [A, B] &= AB - BA \\ &= -(BA - AB) \\ &= -[B, A] \end{aligned}$$

Skew-symmetry is satisfied.

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Lastly we check that $\mathfrak{h}$ satisfies the Jacobi identity.

Let $A, B, C \in H$, then firstly

$$\begin{aligned}
&[A,[B,C]] + [B,[C,A]] \\
&= A[B,C] - [B,C]A + B[C,A] - [C,A]B \\
&= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B \\
&= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB \\
&= ABC + CBA - BAC - CAB
\end{aligned}$$

Then using this result we have,

$$\begin{aligned}
&[A,[B,C]] + [B,[C,A]] + [C,[A,B]] \\
&= ABC + CBA - BAC - CAB + [C,[A,B]] \\
&= ABC + CBA - BAC - CAB + C[A,B] - [A,B]C \\
&= ABC + CBA - BAC - CAB + C(AB - BA) - (AB - BA)C \\
&= ABC + CBA - BAC - CAB + CAB - CBA - ABC + BAC \\
&= 0
\end{aligned}$$

So the Jacobi identity is satisfied and $\mathfrak{h}$ is indeed a Lie algebra.

3 The geodesic equation and the metric

The particular focus of this project was to classify the *geodesics* of the group. A geodesic is a curve that locally minimises the distance between two points in a space. Considering surfaces in $\mathbb{R}^3$, some examples of geodesics are the straight lines on a plane and the great circles on a sphere. For a curve to be a geodesic it must have a geodesic curvature of 0. What this says is that the tangential component of the second derivative vector of the curve must be 0. For an arbitrary Riemannian space this is expressed by the geodesic equation.

$$u_k'' + \sum_{ij} \Gamma_{ij}^k u_i' u_j' = 0 \tag{1}$$

Where $u_k(t)$ are the coordinate functions of the curve and Γ_{ij}^k are the Christoffel symbols.

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The very first step to classifying our geodesics is to equip our space with a metric, which gives us a notion of distance. For this project, a left invariant metric is used. This will allow us to describe a geodesic issuing from an arbitrary point $(x, y, z) \in \mathbb{R}^3$ in terms of geodesics issuing from the origin, as we will then be able to left translate these curves to the point (x, y, z).

To construct a left invariant metric, we will adopt the approach used in [1], using left invariant vector fields. A vector field X is left invariant if the pushforward from $X(x)$ to $X(y)$ gives the same result as left translating from x to y and then evaluating our vector field for any elements x and y in the space. To obtain left invariant vector fields we let $\gamma_1, \gamma_2, \gamma_3$ be curves through the origin at $t = 0$ and we left translate these curves by an arbitrary point (x, y, z) to get the following curves.

$$\begin{aligned}
\gamma_1 &= (x, y, z) \cdot (t, 0, 0) = (x + t, y, z) \\
\gamma_2 &= (x, y, z) \cdot (0, t, 0) = (x, y + t, z + xt) \\
\gamma_3 &= (x, y, z) \cdot (0, 0, t) = (x, y, z + t)
\end{aligned}$$

Differentiating these left translated curves with respect to t gives the left invariant vector fields X_1, X_2 and X_3.

$$X_1 = (1, 0, 0),\ X_2 = (0, 1, x) \text{ and } X_3 = (0, 0, 1)$$

At this point we make a choice to use this set of vector fields as an orthonormal frame. This means we define the inner product on the tangent space such that $\{X_1, X_2, X_3\}$ is an orthonormal basis of the tangent space at any point $(x, y, z) \in \mathbb{R}^3$. However, this basis is not unique, in fact given an arbitrary basis and any inner product, we can construct a new basis $\{Z_1, Z_2, Z_3\}$ that is orthogonal with all the elements the same length, that satisfies the Lie bracket conditions $[Z_1, Z_2] = Z_3$ and $[Z_1, Z_3] = [Z_2, Z_3] = 0$.

To prove this we let $\{X_1, X_2, X_3\}$ be an arbitrary basis for $\mathfrak{h}$. This basis satisfies the properties $[X_1, X_2] = X_3$ and $[X_1, X_3] = [X_2, X_3] = 0$. Then if we choose any inner product on the tangent space, we can define $Y_3 = aX_3$ where $a \in \mathbb{R}$ and $a \neq 0$, then we can use the Gram-Schmidt orthonormalisation process to construct Y_1 and Y_2. Take $Y_1 = bX_1 - cY_3$ and $Y_2 = dX_2 - eY_1 - fY_3$ where $b, c, d, e, f \in \mathbb{R}$ are the relevant coefficients from the Gram-Schmidt process. It can easily be shown that $[Y_1, Y_2] = \frac{1}{a}Y_3$ and $[Y_1, Y_3] = [Y_2, Y_3] = 0$.

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Then $\{\tilde{Y_1}, \tilde{Y_2}, \tilde{Y_3}\} = \left\{\frac{Y_1}{\|Y_1\|}, \frac{Y_2}{\|Y_2\|}, \frac{Y_3}{\|Y_3\|}\right\}$ is an orthonormal basis. Due to the properties of the original basis and the bilinearity of the Lie bracket, we have $[\tilde{Y_1}, \tilde{Y_3}] = [\tilde{Y_2}, \tilde{Y_3}] = 0$.
Also,

$$\begin{aligned}
[\tilde{Y_1}, \tilde{Y_2}] &= \frac{1}{\|Y_1\|\|Y_2\|}[Y_1, Y_2] \\
&= \frac{1}{a\|Y_1\|\|Y_2\|}Y_3 \\
&= \frac{\|Y_3\|}{a\|Y_1\|\|Y_2\|}\tilde{Y_3} \\
&= \frac{1}{\alpha}\tilde{Y_3}
\end{aligned}$$

where $\alpha = \frac{a\|Y_1\|\|Y_2\|}{\|Y_3\|}$. Then we can define a new basis $\{Z_1, Z_2, Z_3\} = \{\alpha\tilde{Y_1}, \alpha\tilde{Y_2}, \alpha\tilde{Y_3}\}$. As we are simply scaling the orthonormal basis we have $[Z_1, Z_3] = [Z_2, Z_3] = 0$.
Then,

$$\begin{aligned}
[Z_1, Z_2] &= \alpha^2[\tilde{Y_1}, \tilde{Y_2}] \\
&= \frac{\alpha^2}{\alpha}\tilde{Y_3} \\
&= \alpha\tilde{Y_3} \\
&= Z_3
\end{aligned}$$

We now have a basis that satisfies the Lie bracket conditions, is orthogonal and all elements are the same length. So as the metric is defined in terms of the inner product of the basis vectors of the tangent space, all the left invariant metrics we can put on the Heisenberg group are equivalent up to scaling by a constant.

Getting back to our basis $\{X_1, X_2, X_3\} = \{(1,0,0), (0,1,x), (0,0,1)\}$,we can express the standard partial derivatives in the x, y, z directions in terms of our basis.

$$\frac{\partial}{\partial x} = X_1,\ \frac{\partial}{\partial y} = X_2 - xX_3 \text{ and } \frac{\partial}{\partial z} = X_3$$

To obtain our metric, we take the inner product of these partial derivatives. Using the orthonormality of our basis, the metric becomes

$$ds^2 = dx^2 + (1 + x^2)dy^2 - 2xdydz + dz^2$$

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equivalently, the matrix of the metric tensor is given by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & -x \\ 0 & -x & 1 \end{pmatrix}$$

The matrix representation is useful as it allows the inverse matrix to computed using standard linear algebraic techniques. The inverse matrix is given as follows,

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1+x^2 \end{pmatrix} \tag{2}$$

4 The Christoffel symbols

The Christoffel symbols of the first kind are defined in the following way:

$$\Gamma_{ij,l} = \frac{1}{2}\Big(\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{lj}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l}\Big)$$

where $(u_1, u_2, u_3) = (x, y, z)$. As the metric only depends on x, the partial derivatives with respect to y and z will automatically be 0. The only nonzero symbols are:

$$\begin{aligned} \Gamma_{22,1} &= -x \\ \Gamma_{23,1} = \Gamma_{32,1} &= \frac{1}{2} \\ \Gamma_{12,2} = \Gamma_{21,2} &= x \\ \Gamma_{13,2} = \Gamma_{31,2} &= -\frac{1}{2} \\ \Gamma_{12,3} = \Gamma_{21,3} &= -\frac{1}{2} \end{aligned}$$

These symbols of the first kind can then be used in conjunction with the matrix inverse to the matrix of the metric tensor (2) to obtain the Christoffel symbols of the second kind by the formula:

$$\Gamma^k_{ij} = \sum_{l=1}^{3} (g^{-1})_{kl} \Gamma_{ij,l}$$

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This gives the following nonzero symbols of the second kind:

$$\begin{aligned}
\Gamma^1_{22} &= -x \\
\Gamma^1_{23} = \Gamma^1_{32} &= \frac{1}{2} \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{2}x \\
\Gamma^2_{13} = \Gamma^2_{31} &= -\frac{1}{2} \\
\Gamma^3_{12} = \Gamma^3_{21} &= \frac{1}{2}(x^2-1) \\
\Gamma^3_{13} = \Gamma^3_{31} &= -\frac{1}{2}x
\end{aligned}$$

5 The geodesics

Using the Christoffel symbols and (1), we obtain the following system of differential equations.

$$x'' - x(y')^2 + y'z' = 0 \tag{3}$$

$$y'' + xx'y' - x'z' = 0 \tag{4}$$

$$z'' + (x^2-1)x'y' - xx'z' = 0 \tag{5}$$

We can see that our equations do not depend on y and z but only their derivatives, so define $u = y'$ and $v = z'$.
Then our equations become:

$$x'' - xu^2 + uv = 0 \tag{6}$$

$$u' + xx'u - x'v = 0 \tag{7}$$

$$v' + (x^2-1)x'u - xx'v = 0 \tag{8}$$

Multiplying (7) by x and then subtracting (8) gives

$$xu' + x'u - v' = (xu - v)' = 0$$

So we must have $xu - v = c$, where c is a constant. Rearranging this for v gives

$$v = xu - c \tag{9}$$

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Now we can eliminate v from equations (6) and (7), which gives

$$x'' - cu = 0 \tag{10}$$
$$u' + cx' = 0 \tag{11}$$

Differentiating (11) gives

$$u'' + x'' = u'' + c^2 u = 0 \tag{12}$$

Now we have two separate cases to consider, $c = 0$ and $c \neq 0$.

Case 1. $c = 0$.

Equations (10) and(11) become:

$$x'' = 0$$
$$u' = 0$$

This gives $x = At + B$ and $u = C$, so $y = Ct + D$ with $A, B, C, D \in \mathbb{R}$. From equation (9) we have $v = xu = ACt + BC$, which gives $z = \frac{1}{2}ACt^2 + BCt + E$. Using the initial conditions $x(0) = y(0) = z(0) = 0$, we get $B = D = E = 0$. So our solutions are:

$$x = At$$
$$y = Ct$$
$$z = \frac{1}{2}ACt^2$$

This solution gives a family of parabolas (which degenerate to lines when $A = 0$ or $C = 0$) that forms a hyperbolic paraboloid $z = \frac{1}{2}xy$.

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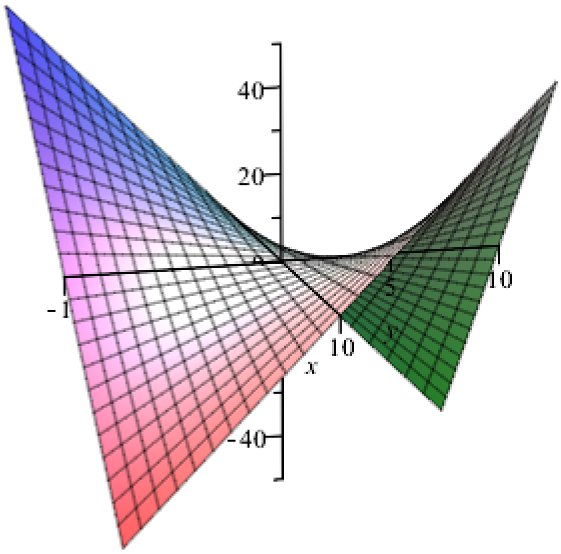

Figure 1: Saddle formed by taking the union of geodesics from Case 1 over all pairs $(A, C) \in \mathbb{R}^2$.

Case 2. $c \neq 0$.

Equation (12) is an ordinary differential equation which has the general solution $u = A\cos(ct) + B\sin(ct)$. As we are finding the geodesics issuing from the origin, we will integrate from 0 to t and use the initial conditions $x(0) = y(0) = z(0) = 0$. Then we get:

$$\begin{aligned} y(t) &= \int_0^t A\cos(ct) + B\sin(ct)\, dt \\ &= \frac{1}{c}[A\sin(ct)\big|_0^t - B\cos(ct)\big|_0^t] \\ &= \frac{1}{c}[B - B\cos(ct) + A\sin(ct)] \end{aligned}$$

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From equation (11) we have $x' = -\frac{1}{c}u' = A\sin(ct) - B\cos(ct)$.
Then we have:

$$\begin{aligned}
x(t) &= \int_0^t A\sin(ct) - B\cos(ct)\ dt \\
&= \frac{1}{c}[-A\cos(ct)\big|_0^t - B\sin(ct)\big|_0^t] \\
&= \frac{1}{c}[A - A\cos(ct) - B\sin(ct)]
\end{aligned}$$

Now, from equation (9) we have

$$\begin{aligned}
v &= xu - c \\
&= \frac{1}{c}[A - A\cos(ct) - B\sin(ct)][A\cos(ct) + B\sin(ct)] - c \\
&= \frac{1}{c}[A^2\cos(ct) + AB\sin(ct) - A^2\cos^2(ct) - 2AB\cos(ct)\sin(ct) - B^2\sin^2(ct) - c^2] \\
&= \frac{1}{c}[I_1 + I_2 + I_3 + I_4 + I_5 + I_6]
\end{aligned}$$

Then we will have $z(t) = \frac{1}{c}\int_0^t [I_1 + I_2 + I_3 + I_4 + I_5 + I_6]\ dt$.
Firstly,

$$\begin{aligned}
\int_0^t I_1\ dt &= \int_0^t A^2\cos(ct)\ dt \\
&= A^2\sin(ct)\big|_0^t \\
&= \frac{A^2}{c}\sin(ct)
\end{aligned}$$

and,

$$\begin{aligned}
\int_0^t I_2\ dt &= \int_0^t AB\sin(ct)\ dt \\
&= -\frac{AB}{c}\cos(ct)\big|_0^t \\
&= \frac{AB}{c}[1 - \cos(ct)]
\end{aligned}$$

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and,

$$
\begin{aligned}
\int_0^t I_3 \; dt &= \int_0^t -A^2 \cos^2(ct) \; dt \\
&= -\frac{A^2}{2} \int_0^t 1 + \cos(2ct) \; dt \\
&= -\frac{A^2}{2} [t\big|_0^t + \frac{1}{2c} \sin(2ct)\big|_0^t] \\
&= -\frac{A^2}{4c} [2ct + \sin(2ct)]
\end{aligned}
$$

and,

$$
\begin{aligned}
\int_0^t I_4 \; dt &= \int_0^t -2AB \cos(ct) \sin(ct) \; dt \\
&= -2AB \int_0^{\sin(ct)} \frac{1}{c} u \; du \\
&= -\frac{AB}{c} \sin^2(ct)
\end{aligned}
$$

and,

$$
\begin{aligned}
\int_0^t I_5 \; dt &= \int_0^t -B^2 \sin^2(ct) \; dt \\
&= -\frac{B^2}{2} \int_0^t 1 - \cos(2ct) \; dt \\
&= -\frac{B^2}{2} [t\big|_0^t - \frac{1}{2c} \sin(2ct)\big|_0^t] \\
&= -\frac{B^2}{4c} [2ct - \sin(2ct)]
\end{aligned}
$$

and finally,

$$
\begin{aligned}
\int_0^t I_6 \; dt &= \int_0^t -c^2 \; dt \\
&= -c^2 t
\end{aligned}
$$

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Now we have,

$$
\begin{aligned}
z(t) &= \frac{1}{c}\int_0^t [I_1 + I_2 + I_3 + I_4 + I_5 + I_6]\, dt \\
&= \frac{1}{c}[\int_0^t I_1\, dt + \int_0^t I_2\, dt + \int_0^t I_3\, dt + \int_0^t I_4\, dt + \int_0^t I_5\, dt + \int_0^t I_6\, dt] \\
&= \frac{A^2}{c^2}\sin(ct) + [\frac{B^2 - A^2}{4c^2}]\sin(2ct) + \frac{AB}{c^2}[\cos^2(ct) - \cos(ct)] - \frac{t}{2c}[A^2 + B^2 + 2c^2]
\end{aligned}
$$

The nonzero case yields two types of geodesics. Taking $A = B = 0$ gives the curve $(0, 0, -c)$ which runs along the z axis. However, when A and B are non zero the geodesics become helical type curves, although not strictly helices due to the trigonometric terms in the z coordinate.

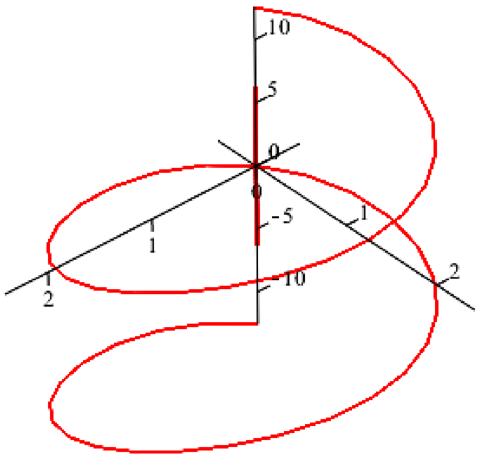

Figure 2: Helical and straight geodesics in Case 2.

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