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The Digraph Lattice

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Abstract

Graph homomorphisms play an important role in graph theory and its applications. For example, the n-colourability of a graph G is equivalent to the existence of a graph homomorphism from G to the complete graph K_n.

Using lattice theory, we re-examine some nice proofs and problems explored by Hell and Nešetřil in their text *Graphs and Homomorphisms*. We investigate the lattices of finite digraphs and finite graphs ordered by homomorphism. We show that the lattice of finite graphs is dense above its unique atom, and that every finite ordered set embeds into both lattices.

1 Motivation

Graph homomorphisms play an important role in graph theory and its applications [5]. For example, the n-colourability of a graph G is equivalent to the existence of a graph homomorphism from G to the complete graph K_n. (In Figure 1 we see the well-known Petersen graph is 3-colourable.) This example is an instance of a *constraint satisfaction problem* (*CSP*). In general, the CSP associated with a finite graph K asks "*which finite graphs have a homomorphism to K?*"

The well-known Dichotomy Conjecture [4], first formulated by Feder and Vardi in 1993, is that the computational complexity of a CSP is either `P` or `NP`-complete. Interest in CSPs is motivated by their wide applications, in areas such as timetabling, constraint programming, database theory and artificial intelligence.

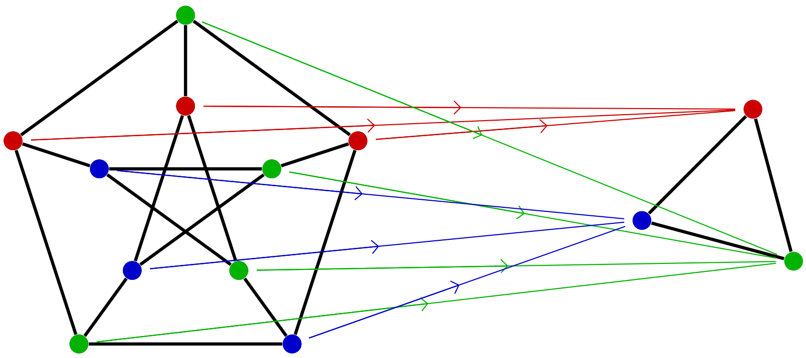

Figure 1: A homomorphism from the Petersen graph to a triangle.

2 Introduction

In this project, we define a *digraph* to be a set with a binary relation on it called the *edge relation*. Then a *graph* is a digraph in which the edge relation is symmetric. We define a quasi-order (that is, a reflexive and transitive binary relation) on the class of all finite digraphs by

$$G \leqslant H \iff \text{there exists a homomorphism } h\colon G \to H.$$

By identifying digraphs that are equivalent under this quasi-order, the class of all finite digraphs becomes a countably infinite ordered set $\mathcal{D}$.

The ordered set $\mathcal{D}$ provides an alternative perspective on some significant results in graph theory. For example, the Four Colour Theorem is equivalent to the statement that the class of finite planar graphs has a maximum element in $\mathcal{D}$ [5].

This project presents a number of important properties of $\mathcal{D}$:

(1) The ordered set $\mathcal{D}$ is a bounded distributive lattice in which the least upper bound of digraphs G and H is their disjoint union $G \dot{\cup} H$, and the greatest lower bound of G and H is their direct product $G \times H$.

(2) The class of finite graphs (that is, symmetric digraphs) yields a sublattice $\mathcal{D}_S$ of $\mathcal{D}$.

(3) Both $\mathcal{D}$ and $\mathcal{D}_S$ are relatively pseudo-complemented lattices: the pseudo-complement of G relative to H is the *exponential digraph* H^G.

(4) Every finite ordered set embeds into $\mathcal{D}_S$, and therefore into $\mathcal{D}$ [6].

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(5) The lattice $\mathcal{D}_S$ is *order-dense* strictly above the bottom element 0, that is, if $0 < G < H$, then there exists a finite graph K with $G < K < H$ [5].

The proof of the graph-theoretic result (5) can be simplified by using a general argument about relatively pseudo-complemented lattices. This result is demonstrative of the aim of the project; that is, to rephrase graph-theoretic problems from an order-theoretic perspective.

We show that order theory can be employed to simplify the proofs of graph-theoretic results presented in Chapter 3 of the text *Graphs and Homomorphisms* by Hell and Nešetřil [5]. We take a more lattice-theoretic approach to the proofs than that of Hell and Nešetřil. This paves the way for future work reframing open graph-theoretic problems as lattice-theoretic problems.

Important properties of digraphs

A digraph is a collection of vertices and directed edges between them.

Definition 2.1. A *digraph* G is a non-empty set $V = V(G)$ of vertices, together with a binary relation $E = E(G)$ on V. If $(u, v) \in E(G)$, then we say that there is an *edge* from u to v. We will only consider finite digraphs.

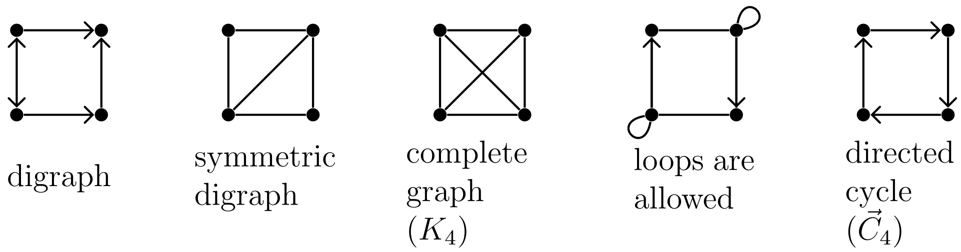


Figure 2: Examples of relevant digraphs.

Definition 2.2. A *graph* is a symmetric digraph.

Figure 2 shows relevant examples of digraphs. A *complete graph* is a symmetric digraph in which all vertices are connected to all other vertices; the complete graph on n vertices is denoted by K_n. A *cycle* can be directed or symmetric; a symmetric cycle on n vertices is denoted by C_n, and when directed, by $\vec{C}_n$. As we consider a digraph to be a set with a binary edge-relation on it, we do not allow, as is sometimes the case, for multiple edges in the same direction between two vertices.

Definition 2.3. A map $h\colon G \to H$ between digraphs G and H is called a *homomorphism* if $(u, v) \in E(G)$ implies $(h(u), h(v)) \in E(H)$, for all $u, v \in V(G)$.

Definition 2.4. The *chromatic number* of a finite graph G, denoted by $\chi(G)$ is the smallest n with $G \to K_n$, or else ∞.

Definition 2.5. The *odd girth* of a finite graph G, denoted by $\mathrm{oddgirth}(G)$ is the smallest odd n such that G has a cycle of length n, or else ∞.

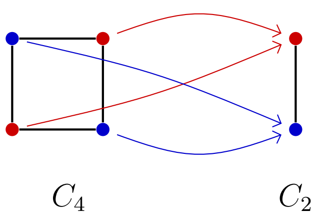
The following important result from graph theory is used without proof. The justification for this is twofold: this is a complex graph-theoretic result, and this project focuses on order theory; and it would have been beyond the scope of this project to do this theorem justice.

Theorem 2.6 (Erdos, 1959)**.** *Let g, k be positive integers, $g \geqslant 3$ odd. Then there exists a connected graph with odd girth at least g and chromatic number at least k [2].*

3 Basic properties of the lattices $\mathcal{D}$ and $\mathcal{D}_S$

For finite digraphs G and H, write $G \to H$ or $G \leqslant H$ if there is a homomorphism from G to H. It requires little work to see that $\leqslant$ is a quasi-order, that is, it satisfies reflexivity and transitivity. The identity map is a homomorphism from G to G. Hence, $G \leqslant G$. So, $\leqslant$ is reflexive. The composition of two homomorphisms is a homomorphism, and so transitivity is satisfied; i.e., if F, G, and H are digraphs, with $F \leqslant G$ and $G \leqslant H$, then $F \leqslant H$. As we have reflexivity and transitivity, the relation $\leqslant$ is a quasi-order on the class of finite digraphs.

The relation $\leqslant$ is not antisymmetric as there exist digraphs G and H where $G \leqslant H$ and $H \leqslant G$ such that $G \neq H$. For example, consider the symmetric cycles C_4 and C_2: We have $C_4 \leqslant C_2$, as shown by the following homomorphism from C_4 to C_2.



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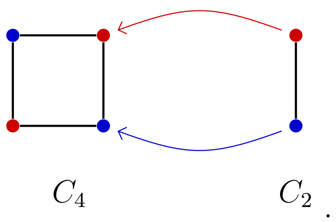
We have $C_2 \leqslant C_4$, as shown by the following homomorphism from C_2 to C_4.



However, C_2 is not the same graph as C_4. So, $\leqslant$ fails antisymmetry. Hence, we have a quasi-order on finite digraphs, but not an order (also known as partial order).

We set $G \equiv H$ if $G \leqslant H$ and $H \leqslant G$. This gives an induced order on the set of equivalence classes of finite digraphs. We shall denote this ordered set by $\mathcal{D}$. Let $\mathcal{D}_S$ denote the sub-ordered set determined by symmetric digraphs.

Now $\leqslant$ is a binary relation between equivalence classes of digraphs. We use $[G]$ to denote the equivalence class containing the digraph G. However, to simplify notation, we will not always distinguish between an equivalence class and a representative of the equivalence class.

Lemma 3.1. *The ordered set $\mathcal{D}$ is bounded.*

Proof. Consider an edgeless digraph G. Any homomorphism from G preserves edges as there are no edges to preserve, and there cannot exist a homomorphism from any digraph that has at least one edge to G. Hence, the class of edgeless digraphs is the bottom of $\mathcal{D}$ (Figure 3).

Figure 3: The bottom and top of $\mathcal{D}$.

Now consider any digraph with a loop. Adjacency of vertices is preserved in any map that directs all vertices to a looped vertex. However, a loop must map to a loop under homomorphism. Hence, the class of digraphs with at least one loop is the top of $\mathcal{D}$ (Figure 3). □

A lattice is an ordered set with a least upper bound (join, denoted by $\vee$) and a greatest lower bound (meet, denoted by $\wedge$) for any pair of elements [1, p. 34].

Lemma 3.2. *Least upper bounds exist in $\mathcal{D}$, with $[G] \vee [H] = [G \dot\cup H]$.*

Proof. Clearly $G \to G \dot\cup H$ and $H \to G \dot\cup H$ by the inclusion map. So, $G \leqslant G \dot\cup H$ and $H \leqslant G \dot\cup H$. It remains to show that $G \dot\cup H$ is the least upper bound of G and H.

Let K be an upper bound for G and H. Then $G \leqslant K$ and $H \leqslant K$ via homomorphisms $\varphi\colon G \to K$ and $\psi\colon H \to K$. So, $G \dot\cup H \leqslant K$ by $\varphi \dot\cup \psi\colon G \dot\cup H \to K$. Hence, $G \dot\cup H$ is the least upper bound for G and H. □

Lemma 3.3. *Greatest lower bounds exist in $\mathcal{D}$, with $[G] \wedge [H] = [G \times H]$.*

Proof. We have $G \times H \to G$ and $G \times H \to H$ by the first and second projections, respectively; i.e., $G \times H \leqslant G$ and $G \times H \leqslant H$. So, $G \times H$ is a lower bound of G and H. It remains to show that $G \times H$ is the greatest lower bound of G and H.

Consider any lower bound K of G and H. Then $K \leqslant G$ and $K \leqslant H$ via homomorphisms $\varphi\colon K \to G$ and $\psi\colon K \to H$. So, $K \leqslant G \times H$ via the natural map $\varphi \sqcap \psi\colon K \to G \times H$ given by $(\varphi \sqcap \psi)(a) := (\varphi(a), \psi(a))$. Thus $G \times H$ is the greatest lower bound of G and H. □

As there exist a meet, by Lemma 3.3, and join, by Lemma 3.2, for each pair of elements in $\mathcal{D}$, we have that $\mathcal{D}$ is a lattice. Furthermore, as the disjoint union of two symmetric digraphs is symmetric, and the direct product of two symmetric digraphs is symmetric, $\mathcal{D}_S$ is a lattice. By Lemma 3.1, we have $\mathcal{D}$ and $\mathcal{D}_S$ are bounded lattices.

Thus we have the following important observation.

Theorem 3.4. *$\mathcal{D}$ is a lattice and $\mathcal{D}_S$ is a sublattice.*

It is natural to ask if $\mathcal{D}$ and $\mathcal{D}_S$ satisfy any special lattice identities.

Lemma 3.5. *$\mathcal{D}$ and $\mathcal{D}_S$ are distributive lattices.*

Proof. A lattice is distributive if

$$(\forall a, b, c \in L)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

[1, p. 86].

Let $F, G, H \in \mathcal{D}$. We have just seen that $[X] \vee [Y] = [X \dot\cup Y]$ and $[X] \wedge [Y] = [X \times Y]$ for any $X, Y \in \mathcal{D}$. So,

$$\begin{aligned}
[F] \wedge ([G] \vee [H]) &= [F] \wedge [G \dot\cup H] \\
&= [F \times (G \dot\cup H)] \\
&= [(F \times G) \dot\cup (F \times H)] \\
&= [F \times G] \vee [F \times H] \\
&= ([F] \wedge [G]) \vee ([F] \wedge [H]).
\end{aligned}$$

Hence, $\mathcal{D}$ is a distributive lattice and consequently so is its sublattice $\mathcal{D}_S$. □

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Lemma 3.6. *$\mathcal{D}_S$, and therefore $\mathcal{D}$, is infinite. In fact, $\mathcal{D}_S$ contains a countably infinite chain.*

Proof. We will show that $K_1 < K_2 < K_3 < \dots$. Let G and H be complete graphs with no loops of k and $k+1$ vertices, respectively. We have $G \leqslant H$ by mapping each distinct vertex in G to a distinct vertex in H (Figure 4). In order to map H to G, however, we must map two vertices of H to one vertex of G. As the graphs are complete, there are edges between every pair of distinct vertices, so to preserve the edge between the vertices mapped to a single vertex we would require a loop in G. Hence, $H \nleqslant G$. □

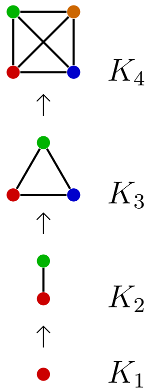

Figure 4: $K_1 \to K_2 \to K_3 \to K_4$.

Lemma 3.7. *In both $\mathcal{D}$ and $\mathcal{D}_S$, an element is join-irreducible if and only if it is not the bottom of the lattice and has a connected representative.*

Proof. By definition, the bottom of a lattice is not join-irreducible. So, let $[X]$ be an element of $\mathcal{D}$ other than the bottom. Assume that X is connected. To prove that $[X]$ is join-irreducible it suffices to show that if $[X] = [G] \vee [H]$ for some finite digraphs G and H, then $[X] \leqslant [G]$ or $[X] \leqslant [H]$. As $[X] = [G] \vee [H] = [G \,\dot{\cup}\, H]$, there exists a homomorphism $\varphi\colon X \to G \,\dot{\cup}\, H$. As X is connected, we must have $\varphi(X) \subseteq G$ or $\varphi(X) \subseteq H$, say the former. Thus, by restricting the codomain of φ to G, we have a homomorphism $\varphi\colon X \to G$. Hence, $[X] \leqslant [G]$, as required. Thus $[X]$ is join-irreducible.

Now assume that $[X]$ is join-irreducible. As X is finite, there exist connected graphs $X_1, \dots, X_n$ such that $X = X_1 \dot{\cup} \cdots \dot{\cup} X_n$. Consequently, $[X] = [X_1 \dot{\cup} \cdots \dot{\cup} X_n] =$

$[X_1] \vee \cdots \vee [X_n]$. As $[X]$ is join-irreducible, we have $[X] = [X_i]$, for some i. So, $[X]$ has a connected representative, namely X_i. □

Lemma 3.8. *In both $\mathcal{D}$ and $\mathcal{D}_S$, every element is a finite join of join-irreducible elements.*

Proof. We shall give the proof for $\mathcal{D}$—in fact, the same proof works for $\mathcal{D}_S$. Let G be a finite digraph. Since the bottom of any lattice is the join of the empty set, we may assume that $[G]$ is not the bottom of $\mathcal{D}$. We have $G = G_1 \dot\cup \cdots \dot\cup G_n$, where each G_i is connected. Thus,

$$[G] = [G_1 \dot\cup \cdots \dot\cup G_n] = [G_1] \vee \cdots \vee [G_n].$$

If G_i is a single vertex with no loop, then $[G_i]$ equals the bottom of $\mathcal{D}$ and so can be removed from the join on the right-hand side of the equation above. What is left will be a join of join-irreducibles by Lemma 3.8. □

We next show that both $\mathcal{D}$ and $\mathcal{D}_S$ are relatively pseudo-complemented, i.e., they are Heyting algebras.

Definition 3.9. The *exponential digraph* H^G is defined to have the vertex set $V(H)^{V(G)}$ and

$$(\varphi, \psi) \in E(H^G) \iff (\forall (a,b) \in E(G))\ (\varphi(a), \psi(b)) \in E(H).$$

For example, it is straightforward to show that $\vec{C}_3^{C_2}$ is as shown in Figure 5.

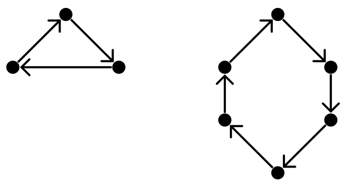

Figure 5: The graph of maps from C_2 to $\vec{C}_3$; i.e., the exponential digraph, $\vec{C}_3^{C_2}$.

Definition 3.10. A *Heyting algebra* H is a bounded lattice where for all $a, b \in H$, there exists a greatest element $x \in H$ such that $a \wedge x \leqslant b$. Then we define $a * b := x$.

Since $\mathcal{D}$ is a bounded lattice, to show $\mathcal{D}$ is a Heyting algebra, we need to find for all $G, H \in \mathcal{D}$ a greatest element F such that $G \wedge F \leqslant H$, i.e.,

$$G * H := \max\{\, F \in \mathcal{D} \mid G \wedge F \leqslant H \,\}.$$

Theorem 3.11. *$\mathcal{D}$ and $\mathcal{D}_S$ are Heyting algebras. In both cases, we have $G * H = H^G$.*

Proof. Clearly it suffices to prove the result for $\mathcal{D}$. Let F, G, and H be finite digraphs.

We wish to show $G * H \equiv H^G$, i.e., $G \wedge F \leqslant H \iff F \leqslant H^G$, i.e.,

(i) if $G \wedge F \leqslant H$ then $F \leqslant H^G$; and,

(ii) if $F \leqslant H^G$ then $G \wedge F \leqslant H$.

to prove (i), let $G \wedge F \leqslant H$, i.e., $G \times F \to H$. So there exists a homomorphism $\varphi : G \times F \to H$, i.e., $((u,a),(v,b)) \in E(G \times F) \Rightarrow (\varphi(u,a), \varphi(v,b)) \in E(H)$.

To prove: $F \leqslant H^G$, i.e., $F \to H^G$, i.e., we have a homomorphism $\psi\colon F \to H^G$. So, if $(a,b) \in E(F)$ then $(\psi(a), \psi(b)) \in E(H^G)$, i.e., for all edges $(u,v) \in E(G)$ we have $(\psi(a)(u), \psi(b)(v)) \in E(H)$ by Definition 3.9.

Let $x \in F$. Define $\psi(x)\colon G \to H$ by $(\forall y \in G)\ \psi(x)(y) := \varphi(y,x)$. We need to show that ψ is a homomorphism. Let $(a,b) \in E(F)$. Let $(u,v) \in E(G)$.

To prove: $(\psi(a)(u), \psi(b)(v)) \in E(H)$. Well, $(\psi(a)(u), \psi(b)(v)) \in E(H) \iff (\varphi(u,a), \varphi(v,b)) \in E(H) \Leftarrow ((u,a),(v,b)) \in E(G \times F) \iff (u,v) \in E(G)$ and $(a,b) \in E(F)$.

As $(a,b) \in E(F)$, and $(u,v) \in E(G)$, we have $((u,a),(v,b)) \in E(G \times F)$. So,

$$\begin{aligned} & (\varphi(u,a), \varphi(v,b)) \in E(H) && \text{as } \varphi \text{ is a homomorphism} \\ \Longrightarrow\ & (\psi(a)(u), \psi(b)(v)) \in E(H) && \text{by the definition of } \psi. \end{aligned}$$

Thus, $(\psi(a), \psi(b)) \in E(H^G)$ as required. Hence $\psi\colon F \to H^G$ is a homomorphism. Thus (i) holds.

To prove (ii), let $F \leqslant H^G$. So, there exists a homomorphism $\psi\colon F \to H^G$. So, $(a,b) \in E(F)$ implies $(\psi(a), \psi(b)) \in E(H^G)$. From Definition 3.9, this gives

$$(\forall (u,v) \in E(G))\ (\psi(a)(u), \psi(b)(v)) \in E(H).$$

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To prove: $G \wedge F \leqslant H$, i.e., there exists a homomorphism $\varphi\colon G \times F \to H$, i.e., if $((u,a),(v,b)) \in E(G \times F)$ then $(\varphi(u,a), \varphi(v,b)) \in E(H)$.

Define $\varphi\colon G \times F \to H$ by $\varphi(y,x) := \psi(x)(y)$. Let $((u,a),(v,b)) \in E(G \times F)$. Then $(u,v) \in E(G)$ and $(a,b) \in E(F)$. So, $(\psi(a)(u), \psi(b)(v)) \in E(H)$ which gives $(\varphi(u,a), \varphi(v,b)) \in E(H)$. Hence, φ is a homomorphism from $G \times F$ to H which gives $G \wedge F \leqslant H$. Thus (ii) holds.

□

4 $\mathcal{D}_S$ is dense above K_2

In this section, we shall prove that the lattice $\mathcal{D}_S$ is dense above its unique atom $[K_2]$. This result is proved in Chapter 3 of Hell and Nešetřil [5]. Our proof separates out the order theory from the graph theory.

Definition 4.1. An ordered set P is *dense* if for all $x, y \in P$ with $x < y$, there exists $z \in P$ with $x < z < y$.

In order to demonstrate density within $\mathcal{D}_S$ we will require an order-theoretic proposition.

Proposition 4.2. *Let* $(P; \leqslant)$ *be an ordered set with* $S \subseteq P$. *Then* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), *where:*

(i) *S is join-dense, i.e.,* $(\forall b \in P)(\exists T \subseteq S)\ b = \bigvee T$*;*

(ii) $(\forall b \in P)\ b = \bigvee({\downarrow} b \cap S)$*;*

(iii) $(\forall a, b \in P)\ b \nleqslant a \Rightarrow (\exists j \in S)\ j \leqslant b$ *and* $j \nleqslant a$.

Proof. Let $(P; \leqslant)$ be an ordered set with $S \subseteq P$. Assume that S is join-dense. Let $b \in P$. Then $b = \bigvee T$ for some $T \subseteq S$. Let $x \in {\downarrow} b \cap S$. Then $x \in {\downarrow} b$, so $x \leqslant b$ by definition. Hence b is an upper bound for ${\downarrow} b \cap S$. Let y be an upper bound of ${\downarrow} b \cap S$, i.e., $(\forall z \in {\downarrow} b \cap S)\ z \leqslant y$. Then $(\forall z \in T)\ z \leqslant y$, as $T \subseteq {\downarrow} b \cap S$. Then, $b \leqslant y$, as $b = \bigvee T$. Thus $b = \bigvee({\downarrow} b \cap S)$. Hence, we have (i) $\Rightarrow$ (ii).

To show (ii) $\Rightarrow$ (i), simply assume $b = \bigvee({\downarrow} b \cap S)$ and then choose $T = {\downarrow} b \cap S$.

Finally, assume (ii) and let $b \nleqslant a$. Suppose that for all $j \in S$, we have $j \nleqslant b$ or $j \leqslant a$, i.e., $j \leqslant b \Rightarrow j \leqslant a$. We know that $b = \bigvee({\downarrow} b \cap S)$. But for all $j \in {\downarrow} b \cap S$ we have $j \leqslant a$, i.e., a is an upper bound of ${\downarrow} b \cap S$. Therefore, $\bigvee({\downarrow} b \cap S) \leqslant a$, i.e., $b \leqslant a$, $\lightning$. Hence, (ii) $\Rightarrow$ (iii), satisfying the last implication of the proposition. □

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Definition 4.3. An element a of a lattice L is called *completely meet-irreducible* if a is not the top of L and $a = \bigwedge S$ implies $a \in S$. Elements that are *completely join-irreducible* are defined dually.

Lemma 4.4. *The bottom of $\mathcal{D}_S$ is completely meet-irreducible and its unique cover is $[K_2]$.*

Proof. We have that $[K_1]$ is the bottom of D_S. So, $K_1 \leqslant K_2$ and $K_2 \nleqslant K_1$.

Consider any graph G with $[K_1] < [G]$. Then $G \geqslant K_1$ and $G \nleqslant K_1$. So G has at least one edge. Hence, $K_2 \leqslant G$. Hence $[K_2]$ is the greatest lower bound of all elements of $\mathcal{D}_S$ strictly above $[K_1]$. Consequently, $[K_1]$ is completely meet-irreducible and $[K_2]$ is its unique upper cover. □

Definition 4.5. An element a of a lattice L is called *completely join-prime* if a is not the bottom of L and $a \leqslant \bigvee S$ implies $a \leqslant s$ for some $s \in S$. Elements that are *completely meet-prime* are defined dually.

Definition 4.6. Let L be a lattice and let $x, y \in L$. Then $\{\uparrow x, \downarrow y\}$ is a *prime-pair partition* of L if $\uparrow x \cap \downarrow y = \varnothing$, and $\uparrow x \cup \downarrow y = L$, from which it follows that x is completely join-prime and y is completely meet-prime.

Lemma 4.7 (See [5, 3.33])**.** *$\{\uparrow[K_2], \downarrow[K_1]\}$ is the only prime-pair partition of $\mathcal{D}_S$.*

Proof. As $[K_1]$ is the bottom of $\mathcal{D}_S$, it follows from Lemma 4.4 that $\{\uparrow K_2, \downarrow K_1\}$ is a prime-pair partition of $\mathcal{D}_S$.

Let F and H be graphs. Assume $F \nleqslant H$ (then $F \not\equiv K_1$) with $F \not\equiv K_2$. As $F \not\equiv K_1$ and $F \not\equiv K_2$, then oddgirth$(F) \neq \infty$; i.e., F is not bipartite. As $F \nleqslant H$, we have H is not the top of $\mathcal{D}_S$, i.e., H does not have any loops, and so $\chi(H) \neq \infty$. As oddgirth$(F), \chi(H) \in \mathbb{N}$, by Theorem 2.6 there exists a finite connected graph G with $\chi(G) > \chi(H)$ and oddgirth$(G) >$ oddgirth(F). Now, χ is order-preserving. So, $\chi(G) > \chi(H)$ implies that $G \nleqslant H$. So, $G \notin \downarrow H$. Also, oddgirth is order-reversing. So, oddgirth$(G) >$ oddgirth(F) implies that $G \ngeqslant F$. So, $G \notin \uparrow F$. Thus $\{\uparrow[F], \downarrow[H]\}$ is not a prime-pair partition of $\mathcal{D}_S$. Hence, $\{\uparrow[K_2], \downarrow[K_1]\}$ is the only prime-pair partition of $\mathcal{D}_S$. □

Lemma 4.8. *Assume that L is a Heyting algebra such that:*

(i) *each element of L is a (possibly infinite) join of join-irreducibles; and*

(ii) *L has no prime pair partitions.*

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Then L is dense.

Proof. Let $a, b \in L$ with $a < b$. To show L is dense, we need to find an $x \in L$ such that $a < x < b$. As $a < b$ there exists a join-irreducible element $j \in L$ with $j \leqslant b$ and $j \nleqslant a$, by Proposition 4.2. Then $\{\uparrow j, \downarrow(b * a)\}$ is not a partition of L, as L has no prime-pair partitions by assumption. We claim that $\uparrow j \cap \downarrow(b * a) = \varnothing$.

Suppose, by way of contradiction, $\uparrow j \cap \downarrow(b * a) \neq \varnothing$. Then there exists a $z \in L$ such that $z \in \uparrow j \cap \downarrow(b * a)$. So, $z \geqslant j$ and $z \leqslant b * a$. Then, by transitivity, $j \leqslant b * a$. By assumption, $j \leqslant b$. Hence, $j \leqslant b \wedge (b * a)$ and $b \wedge (b * a) \leqslant a$ which implies $j \leqslant a$, $\lightning$. Hence, $\uparrow j \cap \downarrow(b * a) = \varnothing$. So, there exists $c \in L$ with $c \notin \uparrow j \cup \downarrow(b * a)$. Define $x := (c \wedge b) \vee a$. Then $a \leqslant x \leqslant b$. (Indeed, $x \geqslant a$ as $x = (c \wedge b) \vee a$, and $c \wedge b \leqslant b$ and $a \leqslant b$ imply $x = (c \wedge b) \vee a \leqslant b$.)

It remains to show $x \neq a$ and $x \neq b$. Suppose $x = a$. Then

$$(c \wedge b) \vee a = a \Rightarrow c \wedge b \leqslant a \Rightarrow c \leqslant b * a.$$

But $c \notin \downarrow(b * a)$, $\lightning$. Hence, $x \neq a$. Now suppose $x = b$, i.e., $b = (c \wedge b) \vee a$. Recall that $j \leqslant b$ and $j \nleqslant a$. So, $j \leqslant c \wedge b \vee a$. Now, j is join-prime and $j \nleqslant a$. So, $j \leqslant (c \wedge b)$. So $j \leqslant c \wedge b \leqslant c$; but $c \notin \uparrow j$, $\lightning$. Hence, $x \neq b$. So, $a < x < b$ as required. Hence L is dense. □

In the following proof we need the easily proved fact that if H is a Heyting algebra, then so is the sublattice $\uparrow a$, for all $a \in H$.

Theorem 4.9 (See [5, 3.30]). *$\mathcal{D}_S$ is dense above $[K_2]$.*

Proof. $\mathcal{D}_S$ is a Heyting algebra, by Theorem 3.11, hence $\uparrow[K_2]$ is a Heyting algebra, with no prime-pair partitions, by Lemma 4.7. Any element of $\mathcal{D}_S$ is a join of join-irreducible elements, by Lemma 3.8. Hence, by Lemma 4.8, $\mathcal{D}_S$ is dense above $[K_2]$. □

5 Embedding ordered sets into $\mathcal{D}_S$

We shall use the following lattice-theoretic result. Here $\wp_{\text{fin}}(\mathbb{N})$ denotes the ordered set of all finite subsets of $\mathbb{N}$.

Lemma 5.1. *If a bounded lattice L has an infinite antichain of join-prime elements, then $\wp_{\text{fin}}(\mathbb{N})$ order-embeds into L.*

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Proof. Let $a_1, a_2, a_3, \ldots$ be pairwise non-comparable join-primes in L. For $i \neq j$, we have $a_i \nleqslant a_j$ and $a_j \nleqslant a_i$. And if $a_i \leqslant x \vee y$ for some $x, y \in L$, then $a_i \leqslant x$ or $a_i \leqslant y$, as each a_i is join-prime. By induction, if X is a finite subset of L and $a_i \leqslant \bigvee X$ then $a_i \leqslant x$ for some $x \in X$; hence, if $a_i \nleqslant x$ for all $x \in X$ then $a_i \nleqslant \bigvee X$.

Define $\varphi\colon \wp_{\text{fin}}(\mathbb{N}) \to L$ by $\varphi(S) := \bigvee_{s \in S} a_s$. (Note that $\varphi(\varnothing) = \bot_L$.) To see that φ is an order-embedding, let $S, T \in \wp_{\text{fin}}(\mathbb{N})$.

To prove: φ is an order-embedding, i.e., $S \leqslant T$ in $\wp_{\text{fin}}(\mathbb{N})$ if and only if $\varphi(S) \leqslant \varphi(T)$ in L. As the order on $\wp_{\text{fin}}(\mathbb{N})$ is $\subseteq$, then we have $S \leqslant T$ equivalent to $S \subseteq T$.

So we need to show:

(i) if $S \subseteq T$ in $\wp_{\text{fin}}(\mathbb{N})$ then $\varphi(S) \leqslant \varphi(T)$ in L, and

(ii) if $\varphi(S) \leqslant \varphi(T)$ in L then $S \subseteq T$ in $\wp_{\text{fin}}(\mathbb{N})$, i.e., if $S \nsubseteq T$ in $\wp_{\text{fin}}(\mathbb{N})$ then $\varphi(S) \nleqslant \varphi(T)$ in L.

(i) Assume $S \subseteq T$. Then $\varphi(S) = \bigvee_{s \in S} a_s \leqslant \bigvee_{t \in T} a_t = \varphi(T)$. So, $\varphi(S) \leqslant \varphi(T)$.

(ii) Assume $S \nsubseteq T$. Choose $s \in S \backslash T$. Then $a_s \nleqslant a_t$, for all $t \in T$. As a_s is join-prime, this gives $a_s \nleqslant \bigvee_{t \in T} a_t$. So, $\varphi(S) \nleqslant \varphi(T)$.

By (i) and (ii), we have φ is an order-embedding. So, if a bounded lattice L has an infinite antichain of join-prime elements, then $\wp_{\text{fin}}(\mathbb{N})$ order-embeds into L. $\square$

Clearly it suffices to prove the following result for $\mathcal{D}_S$ (as $\mathcal{D}_S$ is a sublattice of $\mathcal{D}$). We present a separate proof for $\mathcal{D}$ as it can be proved without resort to the deep graph-theoretic Theorem 2.6.

Theorem 5.2 (See [5, 3.5])**.**

(1) $\wp_{\text{fin}}(\mathbb{N})$ *embeds into both* $\mathcal{D}$ *and* $\mathcal{D}_S$*.*

(2) *Every finite ordered set order-embeds into both* $\mathcal{D}$ *and* $\mathcal{D}_S$*.*

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Proof. (1) To see that the use of Lemma 5.1 is justified for $\mathcal{D}$, consider the prime order directed cycles (Figure 6). We can't construct a homomorphism from a cycle that is smaller than another cycle to that larger cycle, as we would need an edge returning to the first vertex sooner than the cycle would allow. That is to say, once you have mapped the first vertex, all others must follow as these are directed cycles.

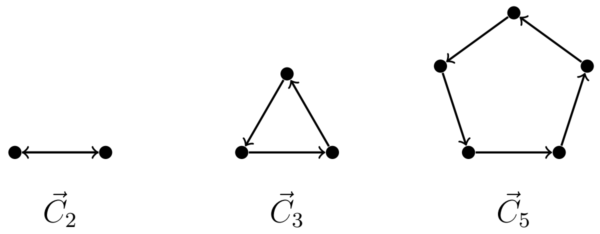

Figure 6: The prime-order cycles of 2, 3, and 5 vertices.

In order to construct a homomorphism from a larger cycle to a smaller cycle, again as they are directed, we require the cycle to wrap around the smaller cycle. We could only do if the size of the larger cycle were a multiple of the smaller cycle, but this is impossible as we are considering cycles of prime order. Hence, the prime-order directed cycles form an infinite antichain of non-comparable elements in $\mathcal{D}$.

To find an infinite antichain of non-comparable elements in $\mathcal{D}_S$, we need to call, once again, upon the graph result outlined in the introduction. By repeated application of Theorem 2.6, there is a sequence of connected graphs $G_1, G_2, G_3, \ldots$ with $\chi(G_1) < \chi(G_2) < \chi(G_3) < \cdots$ and $\text{oddgirth}(G_1) < \text{oddgirth}(G_2) < \text{oddgirth}(G_3) < \cdots$. To check that this is an antichain, assume $G_i \leqslant G_j$. Then, $\chi(G_i) \leqslant \chi(G_j)$ and $\text{oddgirth}(G_i) \geqslant \text{oddgirth}(G_j)$, as χ is order-preserving, and oddgirth is order-reversing. So, $i \leqslant j$ and $i \geqslant j$. Hence, $i = j$. So, $G_i = G_j$. So, there is an infinite antichain of non-comparable elements in $\mathcal{D}_S$.

Both $\mathcal{D}$ and $\mathcal{D}_S$ are distributive, by Lemma 3.5, so all join-irreducible elements are join-prime [1, p. 117]. $\mathcal{D}$ has an infinite antichain of non-comparable, join-prime elements, as does $\mathcal{D}_S$. Hence, by Lemma 5.1, we have $\wp_{\text{fin}}(\mathbb{N})$ order-embeds into $\mathcal{D}$ and $\mathcal{D}_S$.

(2) By (1), it suffices to show that every finite ordered set P embeds into $\wp_{\text{fin}}(\mathbb{N})$. Now, the map $x \mapsto {\downarrow}x$ gives an order-embedding from any finite ordered set, P, to $\mathcal{O}(P)$, where $\mathcal{O}(P)$ denotes the set of down-sets of P ordered by inclusion [1, p. 23]. Clearly $\mathcal{O}(P) \subseteq \wp(\mathcal{P})$, so $\mathcal{O}(P) \hookrightarrow \wp(P)$. Now, $\wp(P)$ is isomorphic to $\wp(\{1, \ldots, n\})$ where $n = |P|$. So, it follows $\wp(P) \hookrightarrow \wp(\{1, \ldots, n\})$, and $\wp(\{1, \ldots, n\}) \subseteq \wp_{\text{fin}}(\mathbb{N})$,

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which implies $\wp(\{1,\ldots,n\}) \hookrightarrow \wp_{\text{fin}}(\mathbb{N})$. So, starting with a finite ordered set, P, we have

$$P \hookrightarrow \mathcal{O}(P) \hookrightarrow \wp(P) \hookrightarrow \wp(\{1,\ldots,n\}) \hookrightarrow \wp_{\text{fin}}(\mathbb{N}).$$

Hence, $P \hookrightarrow \wp_{\text{fin}}(\mathbb{N})$.

□

6 Future directions

Possible future directions for this project are:

(a) review the literature on the lattice-theoretic approach to graph homomorphisms;

(b) give lattice-theoretic proofs of other known graph-theoretic results, such as the extension of the fact that every finite ordered set order embeds into $\mathcal{D}_S$ to the deeper result that every countable ordered set order-embeds into $\mathcal{D}_S$ [6];

(c) establish new properties of the lattices $\mathcal{D}$ and $\mathcal{D}_S$; and,

(d) consider other categories and their homomorphisms from an order-theoretic perspective.

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