How Many Ways are There to Cover a Sphere

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1 Introduction

Hurwitz Numbers enumerate branched covers of sphere. They were introduced in the late 19th century by A. Hurwitz but have recently been studied as they have connections to combinatorics, algebraic geometry and theoretical physics. The aim of my project was to understand a proof of a polynomialy property of the simple Hurwitz numbers and to try to apply the ideas used to find a proof of the generalised spin Hurwitz numbers.

2 Hurwitz Numbers

The *simple Hurwitz number* $H^{\circ}_{g,n}(\mu_1, \ldots, \mu_n)$ is the weighted count of branched covers from a connected genus g surface, with n marked points to $\mathbb{CP}^1$ (the sphere), where each of the n points map to ∞ with branching degree μ_i, and the only other branching is simple branching at m other points on $\mathbb{CP}^1$. The Riemann-Hurwitz formula gives $m = 2g - 2 + n + \sum_{i=1}^{n} \mu_i$.

Proposition 2.1. *$H^{\circ}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{d!} \times \#$ m-tuples$(\sigma_1, \sigma_2, ..., \sigma_m)$ where each $\sigma_i \in S_d$ is a transposition, and $\sigma_1\sigma_2...\sigma_m$ has cycle type $(\mu_1, ..., \mu_n)$ and the subgroup $\langle\sigma_1, \sigma_2, ..., \sigma_m\rangle \subset S_d$ acts transitively on the set $\{1, 2, \ldots, d\}$.*

We say $\langle\sigma_1, \sigma_2, ..., \sigma_m\rangle$ acts transitively on $\{1, 2, \ldots, d\}$ if there is a permutation in

$\langle \sigma_1, \sigma_2, ..., \sigma_m \rangle$ that takes any element of $\{1, 2, \ldots, d\}$ to any other element. This condition is required to ensure the surface covering $\mathbb{CP}^1$ is connected, if we consider possibly disconnected coverings, which we denote $H_{g,n}(\mu_1, \ldots, \mu_n)$ we do not require transitivity.

For example the permutation (123) acts transitively on $\{1, 2, 3\}$, however (12) does not act transitively on $\{1, 2, 3\}$ as there is no way to map 1 to 3 using (12).

A proof of this proposition can be found in [2] and relies on an idea known as monodromy. Typically Hurwitz numbers are defined this way. The advantage of this second definition is that we can calculate the number of these tuples of transpositions using characters from the representation theory of the symmetric group. We will use this to calculate Hurwitz numbers in the infinite wedge space.

3 Infinite Wedge Space

The *Infinite Wedge Space* is an infinite dimensional complex vector space, spanned by the vectors

$$= \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$$

Where $\{s_1 < s_2 < s_3 ...\}$ and each $s_i \in \mathbb{Z} + 1/2$. We also require that for sufficiently large n we have $s_n = s_{n+1}$.

The wedge product $\wedge$ satisfies $\underline{k_1} \wedge \underline{k_2} = -\underline{k_2} \wedge \underline{k_1}$ for $k_1, k_2 \in \mathbb{Z} + 1/2$. Given a set $S = \{s_1 < s_2 < s_3 ...\}$ we denote the vector $\underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$ as v_S.

Usually we only consider a subspace of the infinite wedge space by adding the requirement that $s_i = -i + 1/2$ for sufficiently large i and only consider sets with this property. This is called the zero charge subspace and allows a simple graphical representation of v_S as follows, whenever $s \in S$ we draw a line segment with gradient -1 which we call a downstep, at s. When $s \notin S$ we draw a line segment with gradient 1 at s. For example the set $\{5/2, 1/2, -3/2, -5/2, -9/2, -11/2, \ldots\}$ gives the diagram

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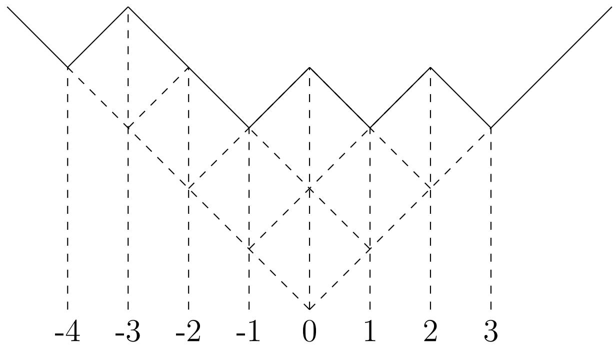
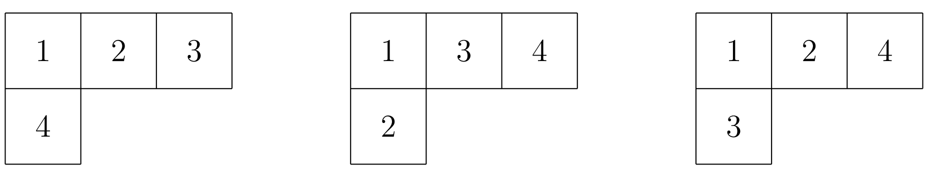


For such a set S we can associate a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k\}$ with $\sum_i \lambda_i = d$ such that $s_i = \lambda_i - i + 1/2$, or $\lambda_i = s_i + i - 1/2$. In the previous example we have $\lambda = \{3, 2, 1, 1\}$. The integer d then gives the total number of 'boxes' in the diagram and λ tells us how they are arranged. Given a partition λ corresponding to a set S we use v_λ to denote the same vector as v_S.

The dimension of a partition, $\dim\lambda$, is defined as the number of standard Young tableaux of shape λ.

A standard Young tableau of shape λ is given by rows of boxes corresponding to the partition, where each of the boxes has one of the numbers from 1 to d and each row (left to right) and column (top to bottom) are in increasing order. For example; $\dim(3, 1) = 3$ as we have



3.1 Operators on the Infinite Wedge Space

We define the operators ψ_k, often called *fermionic operators*, on the infinite wedge space for $k \in \mathbb{Z} + 1/2$ where

$$\psi_k v_S = \underline{k} \wedge \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$$

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$$= \begin{cases} \pm\, v_{S\cup\{k\}} & \text{if } \; k \notin S \\ 0 & \text{if } \; k \in S \end{cases}$$

The sign is given by $(-1)^{\#\, s\in S\,|\, s>k}$ which comes from rearranging the wedge product.

The operator ψ_k^* is the adjoint of ψ_k with respect to the orthonormal inner product, which satisfies $(\psi_k^* v_{S_1}, v_{S_2}) = (v_{S_1}, \psi_k v_{S_2})$.

$$\psi_k^* v_S = \begin{cases} \pm v_{S\setminus\{k\}} & \text{if}\, k \in S \\ 0 & \text{if}\, k \notin S \end{cases}$$

Where the sign is again given by $(-1)^{\#\, s\in S\,|\, s>k}$. Note that ψ_k increases the charge by 1, and ψ_k^* decreases the charge by 1.

The ψ operators satisfy the following anti-commutation relations

$$[\psi_k, \psi_l]_+ = 0$$
$$[\psi_k^*, \psi_l^*]_+ = 0$$
$$[\psi_k^*, \psi_l]_+ = \delta_{k,l}$$

Where the anti-commutator $[a, b]_+$ is $ab + ba$.

Graphically ψ_k changes an upstep to a downstep at k if possible, and ψ_k^* changes a downstep to an upstep at k if possible.

Because the ψ operators change the charge of the vector we usually apply them together to stay within the zero charge subspace.
We define the normally ordered product as follows

$$E_{i,j} := \, :\psi_i \psi_j^*: \, := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0 \\ -\psi_j^* \psi_i, & \text{if} j < 0 \end{cases}$$

As ψ_i and ψ_j anti-commute if $i \neq j$ the normal ordering only matters when $i = j$. Normal ordering is important as it regularises certain sums that arise.

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We also define the *bosonic operators* α_n for $n \neq 0$

$$\alpha_n = \sum_{k \in \mathbb{Z}+1/2} \psi_{k-n} \psi_k^*$$

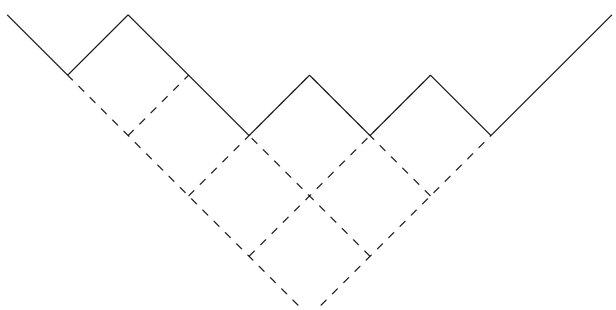
The adjoint of α can be computed in terms of the adjoint of the ψ operators.
We have

$$\begin{aligned} \alpha_n^* &= (\sum_{k \in \mathbb{Z}+1/2} \psi_{k-n} \psi_k^*)^* \\ &= \sum_{k \in \mathbb{Z}+1/2} \psi_k^{**} \psi_{k-n}^* \\ &= \sum_{k \in \mathbb{Z}+1/2} \psi_{k+n} \psi_k^* \\ &= \alpha_{-n} \end{aligned}$$

The α operators satisfy the following commutation relation.

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}$$

We think of α_n as removing a ribbon of n boxes from v_λ and summing over all ways this is possible, and α_{-n} as adding a ribbon of n boxes.
For example starting with $v_{3,2,1,1}$



α_3 will try to remove 3 connected boxes from $v_{3,2,1,1}$, there is only 1 way this is possible which gives $\alpha_3 v_{3,2,1,1} = -v_{1,1,1,1}$ as shown below. The coefficient -1 is due to the sign discussed earlier.

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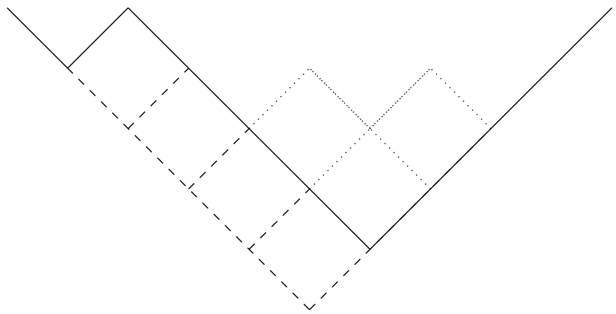

The removed boxes are shown by the dotted lines.

Similarly we could add a box to $v_{3,2,1,1}$ in 4 different ways and we get $\alpha_{-1}v_{3,2,1,1} = v_{4,2,1,1} + v_{3,3,1,1} + v_{3,2,2,1} + v_{3,2,1,1,1}$.

If we apply α_1 d times to the empty partition $v_\emptyset$ we find that

$$\alpha_1^d v_\emptyset = \sum_{|\lambda|=d} \dim\lambda v_\lambda$$

Where $|\lambda| = \sum_i \lambda_i$. We can use this to generalise the concept of dimension to count the number of ways add arbitrary ribbons to $v_\emptyset$. For another partition $\mu = (\mu_1, \mu_2, \ldots \mu_k)$ with $|\mu| = |\lambda|$. χ_μ^λ is the character from the representation theory of the symmetric group. It can be shown using the Murnaghan-Nakayama rule that

$$\prod_{i=1}^n \alpha_{\mu_i} = \sum_{|\lambda|=d} \chi_\mu^\lambda v_\lambda$$

Note that $\chi_{1,\ldots,1}^\lambda = \dim\lambda$.

We can now calculate Hurwitz numbers in the infinite wedge space.

Proposition 3.1. $H_{g,n}(\mu_1, \ldots, \mu_n) = \frac{|C_\mu|}{d!}\langle e^{\alpha_1}\mathcal{F}_2^m \prod_{i=1}^n \alpha_{\mu_i}\rangle$

The *expectation* $\langle A\rangle$ of an operator A is $(v_\emptyset, Av_\emptyset)$.

Proof. Recall the definition of disconnected Hurwitz numbers
$H_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{d!} \times \#$ m-tuples$(\sigma_1, \sigma_2, ..., \sigma_m)$ where each $\sigma_i \in S_d$ is a transposition, and $\sigma_1\sigma_2...\sigma_m$ has cycle type $(\mu_1, ..., \mu_n)$.

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This is given by the Burside formula, using representation theory. We have

$$H_{g,n}(\mu_1,\ldots,\mu_n) = \frac{1}{d!}\sum_{|\lambda|=d}\frac{\dim\lambda|C_\mu|\chi^\lambda_\mu}{d!}f_2(\lambda)^m$$

Where $f_2(\lambda) = \frac{1}{2}\sum_{i=0}^{\infty}[(\lambda_i - i + 1/2)^2 - (-i+1/2)^2]$. Define $\mathcal{F}_2$ such that $\mathcal{F}_2 v_\lambda = f_2(\lambda)v_\lambda$. $|C_\mu|$ denotes the number of permutations in S_d with cycle structure μ.

We previously showed that $\alpha^d_{-1}v_\emptyset = \sum_{|\lambda|=d}\dim\lambda v_\lambda$ and that $\prod_{i=1}^{n}\alpha_{\mu_i} = \sum_{|\lambda|=d}\chi^\lambda_\mu v_\lambda$.

So we have

$$\begin{aligned}(\alpha^d_{-1}v_\emptyset, \mathcal{F}_2^m\prod_{i=1}^{n}\alpha_{\mu_i}v_\emptyset) &= (\sum_{|\lambda|=d}\dim\lambda v_\lambda, \mathcal{F}_2^m\sum_{|\lambda|=d}\chi^\lambda_\mu v_\lambda)\\ &= (\sum_{|\lambda|=d}\dim\lambda v_\lambda, \sum_{|\lambda|=d}\chi^\lambda_\mu f_2(\lambda)^m v_\lambda)\\ &= \sum_{|\lambda|=d}\dim\lambda\chi^\lambda_\mu f_2(\lambda)^m(v_\lambda, v_\lambda)\end{aligned}$$

Then using the property of the adjoint and multiplying through by $\frac{|C_\mu|}{d!^2}$ we have

$$\begin{aligned}\frac{|C_\mu|}{d!^2}(v_\emptyset, \alpha^d_1\mathcal{F}_2^m\prod_{i=1}^{n}\alpha_{\mu_i}v_\emptyset) &= \frac{|C_\mu|}{d!^2}\sum_{|\lambda|=d}\dim\lambda\chi^\lambda_\mu f_2(\lambda)^m(v_\lambda, v_\lambda)\\ &= H_{g,n}(\mu_1,\ldots,\mu_n)\end{aligned}$$

Finally we use the taylor expansion of e^{α_1} and note that only the α_1^d term contributes to the inner product as it is orthonormal. So we have

$$H_{g,n}(\mu_1,\ldots,\mu_n) = \frac{|C_\mu|}{d!}\langle e^{\alpha_1}\mathcal{F}_2^m\prod_{i=1}^{n}\alpha_{\mu_i}\rangle$$

□

Having shown that we can express Hurwitz numbers in the infinite wedge space we can now state the following theorem and give an outline of its proof.

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Theorem 3.2. *The simple Hurwitz numbers satisfy the following polynomiality relation for $(g, n) \neq (0, 1)$ or $(0, 2)$.*

$$H^\circ_{g,n}(\mu_1, \ldots, \mu_n) = m! \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} P_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$$

Where $P_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ is a polynomial in the variables $\mu_1, \mu_2, \ldots, \mu_n$.

This was first proved in 2001 by Ekedahl, Lando, Shapiro and Vainshtein using quite advanced algebraic geometry. This proof gives explicit coefficients for the polynomial and the result is known as the ELSV Formula.

A second proof was given in 2013 by Dunin-Barkowski, Kazarian, Orantin, Shadrin and Spitz using the infinite wedge space. We will give a brief outline of this proof here. See [1] for the full proof.

First we define a genus generating function for the (possibly disconnected) simple Hurwitz numbers.

$$h_\mu(u) = \sum_{g=-\infty}^{\infty} \frac{u^m}{m!} H_{g,n}(\mu_1, \ldots, \mu_n)$$

The sum starts at $-\infty$ as disconnected surfaces can have negative genus. We then use our formula for the disconnected Hurwitz numbers using our infinite wedge space construction. As in [1] we ignore the $|C_\mu|/d!$ term as it is not particularly important. So we have

$$h_\mu(u) = \sum_{g=-\infty}^{\infty} \frac{u^m}{m!} \langle e^{\alpha_1} \mathcal{F}_2^m \prod_{i=1}^{n} \alpha_{\mu_i} \rangle$$

We then bring the sum into the inner product

$$h_\mu(u) = \langle e^{\alpha_1} \sum_{g=-\infty}^{\infty} \frac{u^m \mathcal{F}_2^m}{m!} \prod_{i=1}^{n} \alpha_{\mu_i} \rangle$$

$$= \langle e^{\alpha_1} e^{u\mathcal{F}_2} \prod_{i=1}^{n} \alpha_{\mu_i} \rangle$$

Since $e^{-\alpha_1} v_\emptyset = v_\emptyset$ and $e^{-u\mathcal{F}_2} v_\emptyset = v_\emptyset$ we have

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$$h_\mu(u) = \langle e^{\alpha_1} e^{u\mathcal{F}_2} \prod_{i=1}^{n} \alpha_{\mu_i} e^{-\alpha_1} e^{-u\mathcal{F}_2} \rangle$$

$$= \langle \prod_{i=1}^{n} (e^{\alpha_1} e^{u\mathcal{F}_2} \alpha_{\mu_i} e^{-\alpha_1} e^{-u\mathcal{F}_2}) \rangle$$

It can then be shown after some calculations that

$$e^{\alpha_1} e^{u\mathcal{F}_2} \alpha_{\mu_i} e^{-\alpha_1} e^{-u\mathcal{F}_2} = \frac{u^{\mu_i} \mu_i^{\mu_i}}{\mu_i!} \mathcal{A}(\mu_i, u\mu_i)$$

Where the operator $\mathcal{A}$ is defined as follows

$$\mathcal{A}(a,b) := (\frac{e^{b/2} - e^{-b/2}}{b})^a \sum_{k \in \mathbb{Z}} \frac{a!(e^{b/2} - e^{-b/2})^k}{(a+k)!} \mathcal{E}_k(b)$$

And

$$\mathcal{E}_k(b) := \sum_{i \in \mathbb{Z}+1/2} e^{b(i-k/2)} E_{i-k,i} + \frac{\delta_{k,0}}{e^{b/2} - e^{-b/2}}$$

We can then show that

$$h_\mu(u) = u^d \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \langle \prod_{i=1}^{n} \mathcal{A}(\mu_i, u\mu_i) \rangle$$

The rest of the proof uses a result from Okounkov and Pandharipande [3], that we can express the $\mathcal{A}$ operators as a Laurent series. It is then shown that for fixed powers of u the degree of the μ_i are bounded. As well as there are only negative powers of μ_i for terms in the series that correspond to pairs $(g, n) = (0, 1)$ or $(0, 2)$.

4 Spin Hurwitz Numbers

Spin Hurwitz numbers are a generalisation of Hurwitz numbers that allow for more general branching at the other points on $\mathbb{CP}^1$.

Let μ be a partition of d, $r \geq 1$ be an integer and $m = \frac{2g-2+n+d}{r}$.

Then the *Spin Hurwitz Number* is defined as

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$H^r_{g,n}(\mu_1,...,\mu_n) = \frac{1}{d!}\times$ number of m-tuples$(\sigma_1, \sigma_2, ..., \sigma_m)$ where each $\sigma_i \in S_d$ is a completed $(r+1)$ cycle, and $\sigma_1\sigma_2...\sigma_m$ has cycle type $(\mu_1, ..., \mu_n)$.

After learning about the polynomiality property of Hurwitz numbers the remaining time of my project was spent on trying to find a proof for an analogous result for spin Hurwitz numbers which are conjectured to satisfy the following quasi-polynomial relation by trying to generalise the method used in [1]

Conjecture 4.1.

$$H^{(r)}_{g,n}(\mu_1,\ldots,\mu_n) = m! r^{2g-2+n+m} \prod_{i=1}^{n} \frac{(\mu_i/r)^{\lfloor \mu_i/r \rfloor}}{\lfloor \mu_i/r \rfloor !} P^{(r)}_{g,n}(\mu_1,\mu_2,\ldots,\mu_n)$$

Where $P^{(r)}_{g,n}(\mu_1,\mu_2,\ldots,\mu_n)$ is a quasi-polynomial mod n in the variables $\mu_1,\mu_2,\ldots,\mu_n$.

A Quasi-polynomial (mod n) is a function $f : \mathbb{Z} \to \mathbb{R}$ such that $f(x) = f_i(x)$ for a polynomial f_i when $x \equiv i$ (mod n).

Although we did not find any results during this summer project we plan to continue looking at similar problems during my honours year.

I thoroughly enjoyed the project and had a great experience meeting other maths students at the Big Day In. The Vacation Research Scholarship has helped me experience first-hand what maths research is like and I look forward to continuing my studies in maths in the next few years. I would like to thank my supervisor Dr. Norm Do for his help and support throughout this project. I would also like to thank AMSI and CSIRO for their generous funding and for hosting the Big Day In.

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[2] Combinatorial facets of Hurwitz numbers S.K. Lando, Applications of Group Theory to Combinatorics Taylor and Francis Group, London 2008

[3] A. Okounkov and R. Pandharipande, The equivariant Gromov-Witten theory of P1, arXiv:math/0207233 [math.AG] (2002).