AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

National Collaboration in the Mathematical Sciences

AMSI Vacation Project

Special functions

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1 Introduction

Special functions arise in the study of differential equations which are of particular importance in areas such as applied mathematics. I considered Sturm-Lioville problems as many special functions are solutions to these problems. An important property is that the solutions to Sturm-Liouville problems form an orthomormal basis in a Hilbert space. Also this is a complete basis and so all the results from Fourier theory can be generalised to these classes of special functions. Since Sturm-Liouville problems typically arise when solving partial differential equations by separation of variables, I considered Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ and the wave equation $u_{tt} = u_{xx} + u_{yy} + u_{zz}$ in spherical coordinates.

It is no surprise that solutions involve special functions of the separated variables. When considering a spherical structure and imposing boundary conditions a typical series equations that arises is $\sum_{n=0}^{\infty} a_n P_n(\cos\theta) = f(\theta), 0 < \theta < \pi$. Legendre polynomials are solutions to Legendre's equation which is a Sturm-Liouville problem, the above series equation can be solved by expanding $f(\theta)$ in terms of Fourier-Legendre series. We begin by exploiting orthogonality of Legendre polynomials to obtain $f(\theta) = \sum_{n=0}^{\infty} f_n P_n(\cos\theta)$ where $f_n = \int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta d\theta$. Solution follows by comparing coefficients.

Naturally I progressed to studying more complicated boundary conditions, focusing on mixed boundary conditions. It turns out that solving the above problems but instead with an open spherical structure leads to mixed boundary conditions that instead of a series equation we obtain dual series equations. One of the simplest example of a dual series equation is

$$\sum_{n=0}^{\infty} a_n P_n(\cos\theta) = 1, 0 < \theta < \theta_0, \quad \sum_{n=0}^{\infty} (2n+1) a_n P_n(\cos\theta) = 0, \theta_0 < \theta < \pi$$

where θ_0 is a given constant, coefficients a_n are to be determined.

As I quickly discovered, problems of this form cannot be solved accurately by direct means. Applying orthogonality simply gives an ill-conditioned matrix system. The philosophy when solving ill-conditioned problems reformulate the original problem. For dual series equations there is a method called the Abel transformation method that changes to problem in a way that it gives an infinite matrix system that is well-conditioned.

2 Mixed boundary value problems

I shall briefly introduce the two most basic mixed boundary problems that give rise to dual series equations involving special functions. The dual series equations will be quoted and for further detail on their derivation refer to [1] & [2].

2.1 The open conducting spherical shell

We consider solving Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$.
Applying appropriate mixed boundary conditions, the following dual series equations arises

$$\sum_{n=m}^{\infty} A_n^m P_n^m(\cos\theta) = \sum_{n=m}^{\infty} \alpha_n^m P_n^m(\cos\theta), \qquad 0 < \theta < \theta_0 \tag{2.1}$$

$$\sum_{n=m}^{\infty} (2n+1) A_n^m P_n^m(\cos\theta) = 0, \qquad 0 < \theta < \theta_0 \tag{2.2}$$

where θ_0 is a given constant and A_n^m are coefficients to be determined.

2.2 Plane wave diffraction from a hard spherical cap

We now consider solving Helmholtz equation $(\Delta + k^2)U^t = 0$.
Applying appropriate mixed boundary conditions, the following dual series equations arises

$$\sum_{n=m}^{\infty} i^n (2n+1) \frac{(n-m)!}{(n+m)!} j_n'(ka)\{a_n^m + 1\} P_n^m(\cos\alpha) P_n^m(\cos\theta) = 0, \quad 0 < \theta < \theta_0 \tag{2.3}$$

$$\sum_{n=m}^{\infty} i^n (2n+1) \frac{(n-m)!}{(n+m)!} \frac{a_n^m}{h_n^{(1)'}(ka)} P_n^m(\cos\alpha) P_n^m(\cos\theta) = 0, \quad \theta_0 < \theta < \pi \tag{2.4}$$

where θ_0 is a given constant and a_n^m are coefficients to be determined.

3 Abel transformation method

The dual series examples in sections 2.1 and 2.2 can be viewed as particular examples of the more generally defined dual series

$$\sum_{n=0}^{\infty} \lambda_n(\alpha, \beta; \eta) x_n (1 - r_n) P_n^{(\alpha,\beta)}(t) = F(t), \qquad -1 < t < t_0 \tag{3.1}$$

$$\sum_{n=0}^{\infty} x_n (1 - q_n) P_n^{(\alpha,\beta)}(t) = G(t), \qquad t_0 < t < 1 \tag{3.2}$$

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where coefficients x_n are to be determined, t_0 is a given constant, $F(t)$ and $G(t)$ are given functions, $r_n \to 0$ and $q_n \to 0$, and also $\lambda_n(\alpha, \beta; \eta) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1+\eta)}{\Gamma(n+\alpha+1-\eta)\Gamma(n+\beta+1)}$.

The Abel transformation method uses Abel integral representations of special functions (here Jacobi polynomials) to rewrite the dual series equation into a series equation. Observe that due to the coefficient λ_n (3.1) is a slower converging series that (3.2), so in this process (3.1) is integrated in the process. For details on this transformation process see chapter 2 of [1].

What is important to note with this procedure is that it reduces the problem to an infinite system of linear equations that is well-conditioned under truncation.

4 Numerical solution

In this section I shall simply quote the results obtained by carefully analysing the dual series equations in sections 2.1 and 2.2 as special cases of (3.1) and (3.2). Note that the dual series equations in (2.1) and (2.2) involve the associated Legendre polynomials not the Jacobi polynomials. There are formulae that relate these two polynomials.

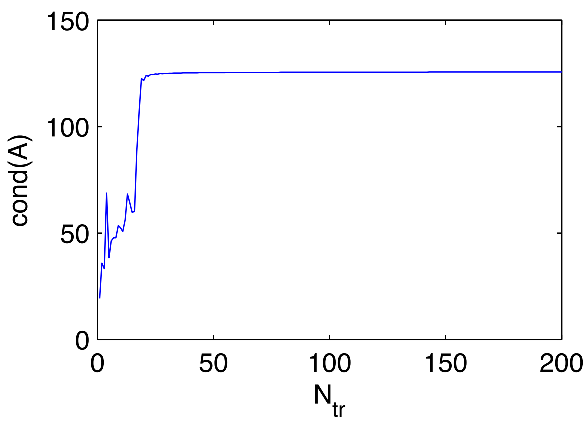
4.1 The open conducting spherical shell

Solution is given $y^m_{p+m} = \sum_{s=0}^{\infty} \hat{\beta}^m_{s+m} \hat{Q}^{(m-\frac{1}{2}, m+\frac{1}{2})}_{ps}(t_0)$ for each $p = 0, 1, 2, 3, \ldots$
where $\{y^m_{p+m}, \hat{\beta}^m_{p+m}\} = \frac{\Gamma(p+m+1)}{\Gamma(p+m+\frac{1}{2})} ||P^{(m-\frac{1}{2}, m+\frac{1}{2})}_p|| \{x^m_{p+m}, \beta^m_{p+m}\}$.
So in the case of the conducting spherical shell an analytic solution exists.

4.2 Plane wave diffraction from a hard spherical cap

Here the infinite system doesn't reduce to an anlytic solution. The details of the matrix system shall be omitted as they can be found in chapter 2.1 of [2]. For illustrative purposes denote this matrix as A. Matlab code can be written to compute values of the above matrix system, some special care is required particularly when considering different values of ka and N_{tr} (truncation number). Note that with special functions such as the Bessel functions, asymptotic behaviour leads to situations where computation involves a zero times an infinity. When using Matlab codes it is appropriate to consider when $N_{tr} \leq 10ka$ as to avoid these asymptotic behaviour.

Consider $ka = 20$, the plot of the condition number of matrix A is shown below



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As the above clearly demonstrates, the matrix A is well-conditioned, in that the condition number is converging for large truncation numbers. We can expect solutions to the truncated matrix system to lose 2 significant figures.

5 Conclusion

As demonstrated solving dual series equations requires different treatment than solving series equations. Infinite matrix systems arising involves computing values of special functions which require some care is needed to ensure that computed values are accurate.

Appendix

Gamma function $\Gamma(z)$ has integral representation

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad z \in \mathbb{R}$$

$$\Gamma(n) = (n-1)!, \qquad n \geq 1$$

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are solutions to the following ODE

$$(1-x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0$$

Legendre polynomials $P_n(x)$, a special case of the Jacobi polynomials (for $\alpha = 0, \beta = 0$) are solutions to the Legendre equation

$$(1-x^2)y'' + 2x)y' + n(n+1)y = 0$$

Associated Legendre polynomials $P_n^m(x)$ are solutions to the more general Legendre equation

$$(1-x^2)\, y'' - 2xy' + \left(n[n+1] - \frac{m^2}{1-x^2} \right) y = 0$$

Bessel functions $J_\nu(z)$ - 1st kind and $Y_\nu(z)$ - 2nd kind are independent solutions to the ODE

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

Hankel functions $H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z)$ is another solution the Bessel's equation.

Spherical Bessel functions take the form

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} K_{\nu+\frac{1}{2}}(z)$$

where $K(z)$ is any Bessel function.

References

[1] Vinogradov et al., 2001, *Canonical problems in scattering and potential theory, part I: Canonical Structures in Potential Theory*, Chapman & Hall/CRC Press.

[2] Vinogradov et al., 2001, *Canonical problems in scattering and potential theory, part II: Acoustic and Electromagetic Diffraction by Canonical Structures* , Chapman & Hall/CRC Press.

[3] Andrews et al., 1999, *Special functions*, Cambridge University Press.