Bernstein's Theorem for Minimal Surfaces

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This project was about minimal surfaces and their properties. Given a fixed closed curve in $\mathbb{R}^3$, the surface with the minimum surface area that has this curve as its boundary is called a minimal surface. All minimal surfaces have the interesting property of zero mean curvature at every point. As the name suggests, mean curvature measures the average amount that a surface curves in any direction (more specifically, it is the average of the maximum and minimum curvatures of all curves on the surface going through the point).

Bernstein's theorem (1915-1917) states that if a minimal surface is defined over the entire x- y plane, then the surface must be a plane. In looking into various proofs of this theorem, I found that there were many different methods for solving the same problem. I studied the proof by Osserman (1966), which used complex analytic techniques to construct a simple proof of the theorem.

I then moved on to looking at the extension of Bernstein's theorem for co-dimension 1 surfaces in higher dimensions (that is, n-dimensional surfaces in $\mathbb{R}^{n+1}$). These surfaces can also be minimal; the mean curvature has a more general meaning in higher dimensional spaces. It was conjectured that a minimal surface in $\mathbb{R}^{n+1}$ defined as a graph over the entire $x_1, x_2, \ldots, x_n$ hyperplane must itself be an n-hyperplane. There were many results relating to this problem published in the 1960s, using the then newly developed techniques of geometric measure theory.

I briefly looked through some of these papers, but many of the methods used required extensive knowledge of measure theory, in which I had very little background. There was one paper, however, that contained a section which used more classical methods to prove a very fundamental result. This paper was the focus of the remaining 4 weeks of my project.

It had been shown in a previous paper that proving the extension of Bernstein's theorem in $\mathbb{R}^{n+1}$ is equivalent to showing that there are no area-minimising cones in $\mathbb{R}^n$. It was also found that there were no minimal cones in 7 and less dimensions. Bombieri, De

Giorgi, and Giusti's paper in 1969 showed that the cones defined by $x_1^2 + x_2^2 + \cdots + x_m^2 = x_{m+1}^2 + x_{m+2}^2 + \cdots + x_{2m}^2$ for $m \geq 4$ have the least area and are therefore minimal surfaces. They also used this result to construct minimal surface graphs that were not hyperplanes in each dimension greater than 8. The result of this paper was a complete solution to the extension of Bernstein's theorem: it is true for up to 8 dimensions, but false for 9 and higher dimensions. Another interesting result is that minimal surfaces in 8 or more dimensions can have singularities, which are points at which a normal vector cannot be assigned to the surface (there is no direction perpendicular to the surface at the point).

The CSIRO Big Day In was a very interesting experience where I learned about many other areas of mathematics from the other speakers. I also gained skills in using the beamer package for LaTeX while preparing my own presentation.

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