Geometry and Topology of 3-manifolds

Richard Hughes

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Abstract

Low-dimensional topology is an extremely rich field of study, with many different and interesting aspects. The aim of this project was to expand upon work previously done by I. R. Aitchison and J.H. Rubinstein on 3-manifolds admitting a polyhedral metric of non-positive curvature.

The primary focus was on manifolds with cubings of non-positive curvature, and in particular the open question of whether or not every 3-manifold admitting such a cubing bounds a 4-manifold admitting such a cubing. Investigation was done using combinatorial techniques, ideas from cobordism theory and surgery theory, hyperbolic geometry and the theory of Coxeter groups.

1 Introduction

The concept of inflicting a polyhedral metric of non-positive curvature on 3-manifolds was first introduced by Rubinstein and Aitchison in [1]. Many interesting examples were dealt with there and for a several years there was a burst of activity on the topic, however there has been little recent work and many questions remain open for exploration.

For this project, the guiding question was whether or not a cubed structure on a 3-manifold would carry over to a 4-manifold for which it is the boundary. Various techniques were drawn upon – cobordism theory was a natural choice, as was the use of Coxeter groups. Canonical immersed hypersurfaces were also studied as a potential constructive technique for cubed 4-manifolds, and Andreev's Theorem [5] was used to define structures admitting a natural cubing.

2 Cubings of non-positive curvature

2.1 Definition and examples

2.1.1 Polyhedral metrics of non-positive curvature

We begin by defining the a polyhedral metric of non-positive curvature on a surface, as per [1], as follows:

Definition 2.1. *A polyhedral metric of non-positive curvature on a close orientable surface is a metric which is locally Euclidean except at a finite number of points, where the dihedral angle is greater than* 2π.

In [1], Aitchison and Rubinstein describe a polyhedral metric on the closed orientable surface of genus 2 (denote this Σ_2), which I will here make visually clear. We can describe Σ_2 as an octagon with sides identified by the word $aba^{-1}b^{-1}cdc^{-1}d^{-1}$, as below in Figure 1.

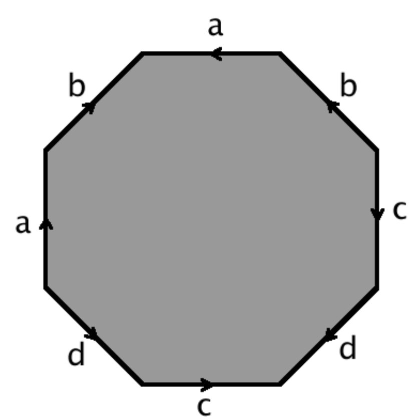

Figure 1: Planar diagram of a genus 2 surface

We then bisect the octagon four times through the centre of each edge to obtain eight quadrilaterals. Treat each of these quadrilaterals as a square as in Figure 2: then the dihedral angle around the central vertex is $8 \times \frac{\pi}{2} = 4\pi$, and the same for the outer vertex.

We can now apply the Gauss-Bonnet Theorem, i.e.

$$\begin{aligned} \int_{\Sigma_2} (\text{Gaussian Curvature}) dA &= \int_{\Sigma_2} \delta(\text{vertex}) \times (\text{Local curvature at each vertex}) dA \\ &= 2 \times \text{Local curvature at each vertex} \\ &= 2\pi \times \chi(\Sigma_2) \\ &= 2\pi(2-2g) \\ &= -4\pi \end{aligned}$$

This gives curvature of -2π at each vertex, and thus our example is made clear.

With this idea under our belt, we move on to the more general definition from [1].

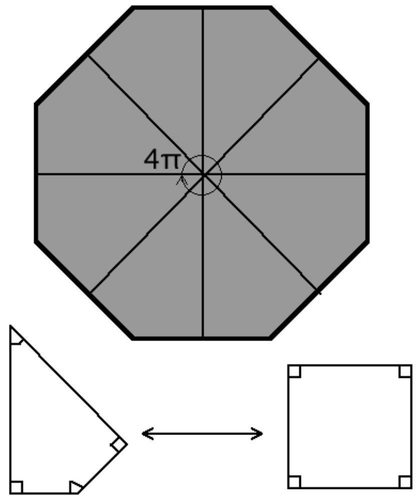

Figure 2: Squaring on the genus 2 surface (with squares made explicit)

Definition 2.2. *M has a polyhedral metric with non-positive curvature if every closed embedded geodesic loop in $lk(Q)$ has length at least 2π for every face Q, assuming that the metric on $lk(Q)$ is scaled to have curvature one at the locally spherical points.*

We won't bother ourselves too much with the details of this definition, except to mention that the construction we will describe is designed to satisfy it.

2.1.2 Cubings with non-positive curvature

A particularly nice combinatorial approach to building 3-manifolds with this metric is described in [1].

Suppose that our 3-manifold M is constructed from a finite collection of regular Euclidean cubes of the same size via identification of faces. Then if the following two conditions hold, M has a polyhedral metric of non-positive curvature:

1. Each edge must belong to at least four cubes.
2. At each vertex, certain local identifications of edges must not occur (i.e. two edges of a single face must not be identified with each other in a neighbourhood of the vertex). Details can be found in [1].

The simplest and most obvious example of a 3-manifold that satisfies these conditions is the 3-torus obtained by identifying opposite faces of a single cube in an orientation preserving manner. Verification of the conditions is a reasonably simple exercise for this example.

2.2 Some rather pleasant properties

2.2.1 Cartan-Hadamard Theorem

In Riemannian geometry, the Cartan-Hadamard Theorem is a result that gives us some rather interesting facts about manifolds with non-positive sectional curvature. The statement of the theorem presented below is from [2]; there one can find a detailed discussion of the theorem and its proof (something we have no space to discuss here).

Theorem 2.1. *(Cartan-Hadamard Theorem). Let M be complete and $K_M \leq 0$ (the sectional curvature). Then for any $p \in M$, $exp_p : M_p \to M$ is a covering map. Hence the universal covering space of M is diffeomorphic to $\mathbb{R}^n$. Hence the homotopy groups $\pi_i(M)$ vanish for $i > 1$.*

This statement is adapted by [1] to give a nice, simple combinatorial version of the Cartan-Hadamard Theorem:

Theorem 2.2. *(Combinatorial Cartan-Hadamard). If an n-manifold M^n has a polyhedral metric with non-positive curvature, then the universal cover of M^n is diffeomorphic to Euclidean $\mathbb{R}^n$.*

Thus any 3-manifold we can construct with our cubing conditions has a nice, known universal cover; moreover, any 3-manifold with cubing is irreducible (every embedded $(n-1)$-sphere bounds an embedded n-ball. [1]

2.2.2 Eilenberg-MacLane property

Another friendly property of our cubed 3-manifolds is that they are so called *Eilenberg-MacLane spaces*. A space X is an Eilenberg-MacLane space of type $K(G, n)$ if the homotopy group $\pi_n(X)$ is isomorphic to G and all other homotopy groups are trivial. Such spaces have important connections with other groups associated with X, in particular the n-th singular cohomology group $H^n(X; G)$. More information can be found in Hatcher's book on Algebraic Topology [3].

In particular, a 3-manifold admitting a non-positive cubing is of type $K(G, 1)$; that is to say, the only (possibly) non-trivial homotopy group is the fundamental group. This follows directly from the combinatorial Cartan-Hadamard theorem.

2.3 An interesting question

This brings us to the question at the heart of our present foray into the world of 3-manifolds. It has been known since the fifties that every 3-manifold bounds a 4-manifold. A proof and discussion of the orientable case can be found in [8] (see [9] ch. 4 for information on Stiefel-Whitney numbers; the general case follows since all Stiefel-Whitney numbers vanish for 3-manifolds). However, there are still open questions regarding the properties any particular 4-manifold may inherit from its bounding 3-manifold. We are guided, then, by the following rather reasonable idea:

Open Question. *Does every 3-manifold with cubing of non-positive curvature bound a 4-manifold with cubing of non-positive curvature?*

For the rest of this report, investigating this question will be our primary goal (in some form or other).

3 Canonical hypersurfaces

3.1 Initial idea

Given an n-manifold W with (hyper-)cubing, we can construct the canonical hypersurface as follows: For each n-dimensional hypercube, build the $(n-1)$-dimensional hypersurfaces that each bisect this hypercube (there ought to be n of these). Then, glue this surface together using the natural identifications inherited from the cubing.

There is no guarantee that this hypersurface will be embedded (i.e. it may have singularities). However, we can treat this hypersurface as the image of a well-behaved manifold M under some immersion $\Phi : M \to W$.

In the case of a 3-manifold, we take each cube and bisect it three times (parallel to each face). We then take the natural gluing inherited from the face identifications of the cubing to obtain our canonical surface.

Note that there is no guarantee that the canonical surface will be either connected or disconnected.

3.2 Separable cubings

We will discuss here a special class of cubings, which are well behaved.

Definition 3.1. *A* separated cubing *on a* 3*-manifold is a cubing of non-positive curvature which admits a* 3*-colouring on each cube that is preserved under face and edge identification (see Figure 3).*

By restricting to this class of cubings, it should be clear that the canonical surface of the manifold admits the same 3-colouring (as in Figure 4). In fact, the canonical surface for the manifold will be disconnected, and will consist of three coloured components which are each separately embedded in the 3-manifold.

By examining how the surfaces intersect each other, and treating intersections as boundary components, we can construct extremely simple representative canonical surfaces.

3.2.1 Simple canonical surfaces

The upshot is as follows: Given a coloured surface Σ and curves of two different colours, we want to construct surfaces that have the curves as boundaries, and have non-intersecting arcs of intersection.

So, suppose we have our surface Σ and curves of two colours (say blue and green) on that surface that intersect in a certain number of places. We want

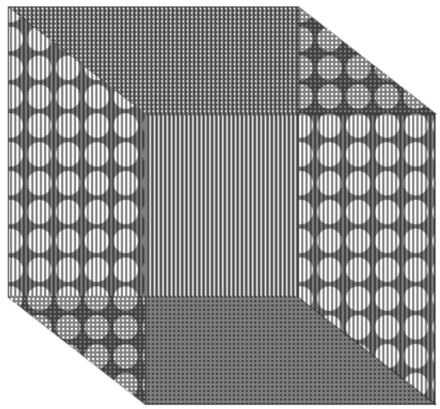

Figure 3: Cube with 3-colouring

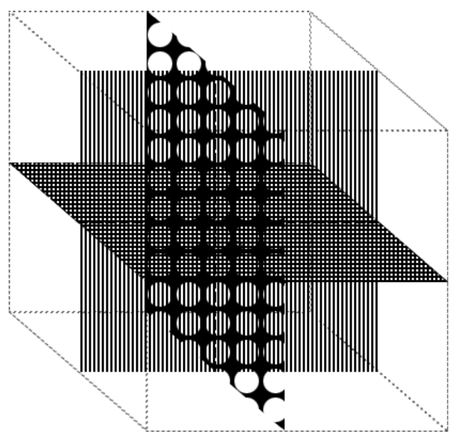

Figure 4: Coloured canonical surfaces of a separated cube

to describe a way to construct a blue surface (Σ') intersected by green arcs. Once the is done, a green surface (Σ'') with blue arcs will follow from the same procedure.

First, for each blue loop on Σ, assign a boundary component on Σ' (the surface under construction).

Intersections of green loops with the boundary components can take a few forms. First, note that the loops of each boundary component can be dealt with separately, thus reducing the complexity of the problem. See Figure 5 for an illustration of this.

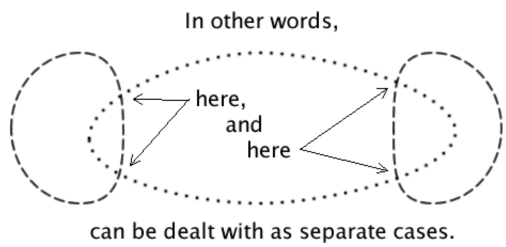

Figure 5: Separated intersected loops on Σ

So, by determining the local structure of Σ' near one boundary component at a time, we can fully determine a structure for Σ'.

Note that here we are going to consider the 'simplest' possible structure for our canonical surfaces – unless absolutely required (which we shall show does not occur), we want our surfaces to be spheres with some number of boundary components. It is also important to remember than the green and blue lines never self intersect on Σ, thanks to the 3-colour condition.

Now, suppose a single green loop intersects a blue loop on Σ more than twice. See Figure 6 for an illustration.

Figure 7 below demonstrates that the surface structure does not need to be changed in this situation; the structure is the same as if two loops had separately intersected the boundary loop on Σ.

It should also be fairly obvious that extra points of intersection $(6, 8, \ldots, 2n, \ldots)$ will not change this structure – the required construction is an simple extension of Figure 7.

Finally we consider the situation where nested green loops intersect a blue boundary on Σ (Figure 8).

We can construct arcs on the surface as follows in Figure 9. This makes clear that no change need be made to the structure of Σ' – it remains a sphere with boundaries.

This exhausts all the possible types of intersections allowed given the 3-colour constraint on our original 3-manifold. Thus, the simplest structure of Σ' is just a sphere with boundaries.

Now, finally, for each surface Σ, Σ' and Σ'' we have *two* intersecting colours to deal with. For instance, for Σ' we have conditions determined by:

1. Blue/green intersections on Σ.

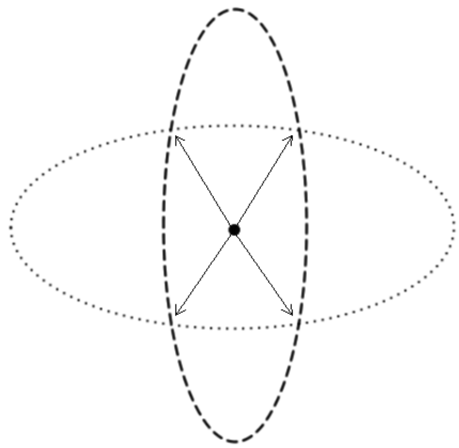

Figure 6: Double intersected loop on Σ

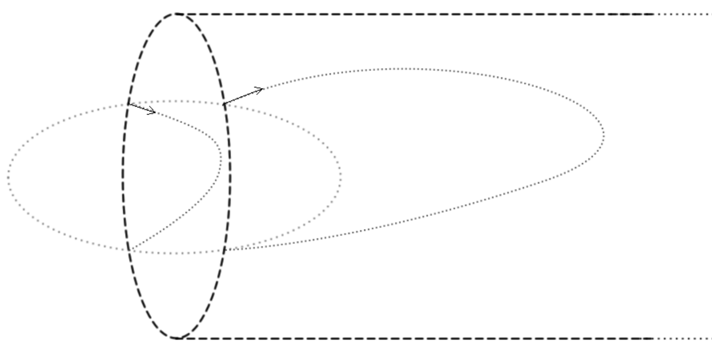

Figure 7: Surface and arc construction from the double intersected loop on Σ

2. Blue/red intersections on Σ''.

Let the sphere with n boundary components be denoted S^2_n. Then if we have

- $\Sigma'_I = S^2_n$ satisfying condition 1
- $\Sigma'_{II} = S^2_m$ satisfying condition 2

it should be clear that

$$\Sigma' = \Sigma'_I \# \Sigma'_{II} = S^2_n \# S^2_m \cong S^2_{n+m}$$

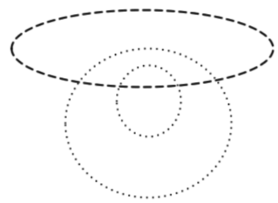

Figure 8: Nested intersection of loops on Σ

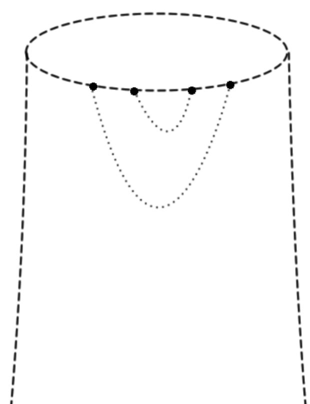

Figure 9: Arc construction for the nested intersection

will simultaneously satisfy both conditions 1 and 2. A quick description of how to see this is as follows:

1. Choose a section of Σ'_I and a section of Σ'_{II} *not* contained in the interior of any loop or arc-boundary segment.

2. Connect sum via the boundaries of these two segments.

3. Note that if we had chosen our segments in a problematic location we could easily have deformed our arcs around them to remove the issue.

Thus we have determined the simplest structure for our canonical surface! Unfortunately, while this construction is somewhat interesting, it does not suggest a clear way forward in terms of reconstructing our 3-manifold.

3.3 An open question regarding canonical surfaces

So far all of the processes described have taken us from an n-manifold to an $(n-1)$-manifold. If we wish to determine whether or not cubed 3-manifolds bound cubed 4-manifolds, we will need to reverse this procedure. This leads us to the following open question.

Open Question. *Given a* 3*-manifold* M *with cubing of non-positive curvature, when can* M *be immersed as a canonical hypersurface for some* 4*-manifold with cubing of non-positive curvature?*

The key to answering this question may lie in the independently interesting 2-dimensional analogue.

Open Question. *Given a surface with squaring of non-positive curvature, when is it possible to immerse our squaring as a (possibly singular) canonical surface inside of a cubed* 3*-manifold?*

This is equivalent to the following question.

Open Question. *Given a surface with squaring of non-positive curvature, when can the squares be given (i) a label* $\mathcal{C}_i$*, (ii) a label* $\{A, B, C\}$ *and (iii) a 'path' of edge identifications such that this assignment is consistent with a cubed* 3*-manifold where the cubes are labelled* $\mathcal{C}_i$*, the bisecting hyperplanes are labelled* $\{A, B, C\}$*, and the 'path' of edge identifications determines the gluing on the faces.*

One obvious example of a necessary condition is that the number of squares in the squaring be a multiple of three.

4 Cobordism theory

4.1 Background

Cobordism theory was originally developed in 1954 by René Thom [4], an achievement for which he would later win the Fields Medal. At the heart of the theory is a rather elegant idea:

Definition 4.1. *Two* n*-dimensional manifolds,* M *and* N *are called* cobordant *if they can be embedded (call the embeddings* i *and* j*) in an* $(n+1)$*-dimensional manifold* W *such that* $\partial W = i(M) \sqcup j(N)$*.*

Less formally, we say that two n-manifolds are cobordant if together they disjointly form the boundary of an $(n+1)$-manifold.

As an extremely trivial example, consider two points on the real line $\{a\}$ and $\{b\}$. If we think of these as 0-manifolds, then the 1-manifold given by the interval $[a, b]$ is a cobordism between them.

4.2 A surgical approach to our guiding question

Another way to approach our problem from the point of view of cobordism theory through the technique of performing surgery on manifolds. When surgery is performed on a manifold M to produce a manifold M', the trace of the surgery gives a cobordism.

Roughly, this corresponds to attaching handles to the boundary of our manifold, and crossing the result with the unit interval so that we 'start' at one end with M and 'finish' at the other end with M'. As an example, we can visualise the trace defining a cobordism between one and two copies of S^1 as a pair of pants (see Figure 10).

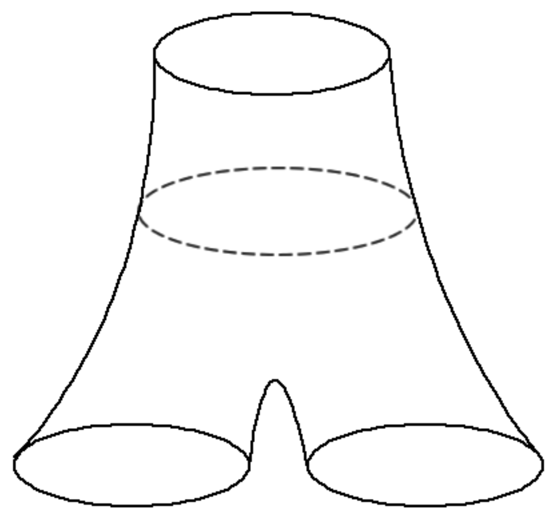

Figure 10: Pair of pants as a cobordism between one and two copies of S^1

This suggests a potential method that could be used to answer our question in the general case, or otherwise construct specific examples: use surgery to reduce (or construct) arbitrary 3-manifolds with cubing to (from) a finite set of cubed 3-manifolds known to bound cubed 4-manifolds, in such a way that the cobordisms preserve the non-positively cubed structure.

In order to prove the conjecture in this way, a number of steps would be required:

1. Determine when surgery on a cubed 3-manifold M will result in a cubed 3-manifold N.

2. Determine when the cobordism between M and N is a cubed 4-manifold W.
3. Show that when two cubed 4-manifolds are glued along boundary components, the result is a cubed 4-manifold.
4. Describe a method to reduce an arbitrary cubed 3-manifold to one of a finite collection of cubed 3-manifolds (using the surgeries allowed given the above conditions).
5. Prove that each of the cubed 3-manifolds in this finite set bounds a 4-manifold with cubing of non-positive curvature.

Such a project would be extremely non-trivial; whether it is even possible to do this is an interesting question. However if this is possible it would be an extremely neat way to prove the conjecture.

5 Coxeter group constructions

5.1 Background

5.1.1 Coxeter groups

First, a definition:

Definition 5.1. *A* Coxeter group *is an abstract group with presentation*

$$\langle r_1, r_2, \ldots, r_n | (r_i r_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$ *and* $m_{ij} \geq 2$ *for* $i \neq j$*. If* $m_{ij} = \infty$*, then there is no relation of the form* $(r_i r_j)^{m_{ij}}$*.*

Coxeter groups can be thought of as a generalisation of reflection groups; each generator r_i of the group represents a reflection across some hyperplane $\mathcal{H}_i$, and each relation $(r_i r_j)^m$ corresponds to the hyperplanes $\mathcal{H}_i$ and $\mathcal{H}_j$ meeting at an angle of $\frac{\pi}{m}$.

Thought of in this way, it is easy to see why each element satisfies the relation $r_i^2 = 1$ (since reflecting twice brings us back to our original position). Similarly, the relations $(r_i r_j)^m$ can be understood as saying that $r_i r_j$ is a rotation of $\frac{2\pi}{m}$ around the intersection of our two hyperplanes.

5.1.2 Andreev's Theorem

A theorem that ought to be mentioned here (its usefulness will become apparent shortly) is Andreev's theorem.

Andreev's theorem is a complete classification of compact hyperbolic polyhedra having non-obtuse dihedral angles. I do not have anywhere near enough space to discuss the proof or details of the theorem here; an excellent reference for that is [5], in which the authors give background, state, and prove the theorem (also correcting a mistake in the original proof).

For our purposes a statement of the proof, along with some basic properties of non-obtuse hyperbolic polyhedra, will suffice:

Theorem 5.1. *(Andreev's Theorem). Let C be an abstract polyhedron with more than four faces, and suppose that non-obtuse angles α_i are given corresponding to each edge e_i of C. There is a compact hyperbolic polyhedron P whose faces realize C with dihedral angle α_i at each edge e_i if and only if the following five conditions all hold:*

1. *For each edge e_i, $\alpha_i > 0$.*
2. *Whenever three distinct edges e_i, e_j, e_k meet at a vertex, then $\alpha_i + \alpha_j + \alpha_k > \pi$.*
3. *Whenever Γ is a prismatic 3-circuit intersecting edges e_i, e_j, e_k, then $\alpha_i + \alpha_j + \alpha_k < \pi$.*
4. *Whenever Γ is a prismatic 4-circuit intersecting edges e_i, e_j, e_k, e_l, then $\alpha_i + \alpha_j + \alpha_k + \alpha_l < 2\pi$.*
5. *Whenever there is a four sided face bounded by edges e_1, e_2, e_3, e_4, enumerated successively, with edges e_{12}, e_{23}, e_{34}, e_{41} entering the four vertices (edge e_{ij} connects the ends of e_i and e_j), then $\alpha_1 + \alpha_3 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi$, and $\alpha_2 + \alpha_4 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi$.*

Furthermore, this polyhedron is unique up to isometries of $\mathbb{H}^3$.

The following facts about such polyhedra will also be useful to keep in mind:

1. A vertex of a non-obtuse hyperbolic polyhedron P is the intersection of exactly three faces.
2. For a cell complex C whose faces correspond to the faces of P, each edge of C belongs to exactly two faces.
3. A non-empty intersection of two faces is either an edge or a vertex.
4. Each face contains not fewer than three edges.

Details on these properties can be found in [5].

5.2 Relationship with cubings

With these facts at hand, it should be clear that there is a 'natural' way to divide up non-obtuse hyperbolic polyhedra into cubes. Given that each vertex is the endpoint of three distinct edges, we can treat this as one corner of a cube. The other edges are then built from the midpoint of the given edges to the centre of the faces those edges belong to, then finally from the centre of the faces to the centre of the polyhedron. The gluing conditions for our cubing should follow directly from the conditions of Andreev's Theorem. A (non-hyperbolic, yet illustrative) example is given in Figure 11.

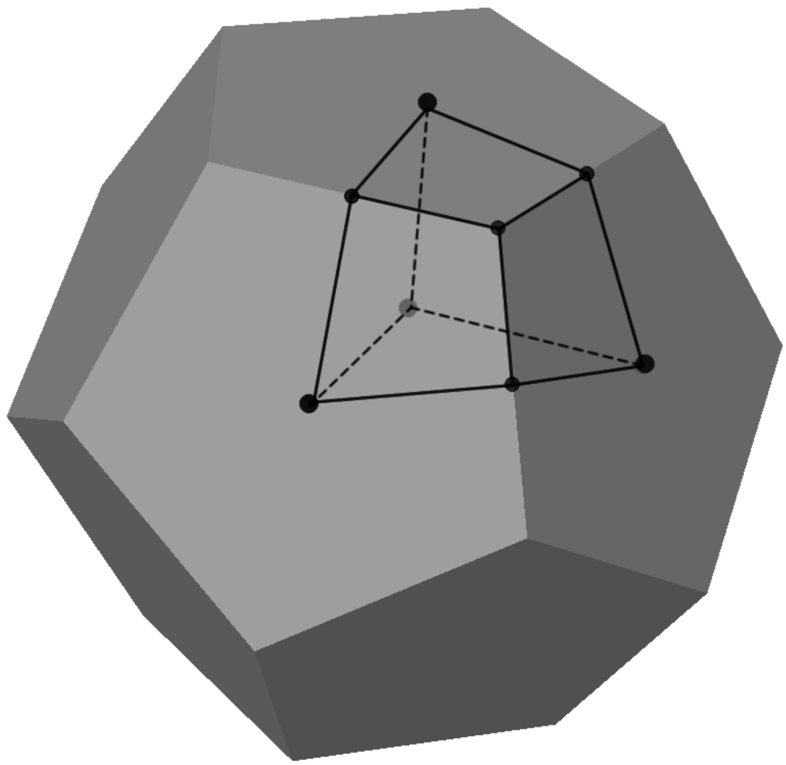

Figure 11: Constructing a cubing on a dodecahedron

Our hope is that, by using the constructive method described below, we can build 4-manifolds which have Andreev's polyhedra as faces, which may allow us to construct a cubing on the 4-manifold.

5.3 Description of constructive method

We begin with a polytope $\mathcal{P}$, whose cell faces are totally geodesic hypersurfaces (i.e. any geodesic that is tangent to the hyperplane lies in the hyperplane) in whichever geometry we are interested in ($\mathbb{R}^n$, $\mathbb{H}^n$, S^n – we will denote an arbitrary geometry as $\mathbb{G}^n$).

We want the angles between the faces of our polytope to be of the form $\frac{\pi}{n}$, for $n \in \mathbb{Z}$ with $n \geq 2$. With such a condition, we can reflect $\mathcal{P}$ across its faces to obtain a tiling of our geometry. Two examples of a tiling for $\mathbb{R}^2$ are shown below in Figures 12 and 13.

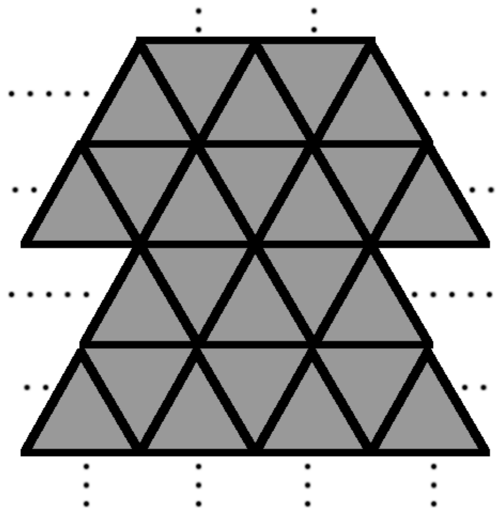

Figure 12: A tiling of the plane with equilateral triangles

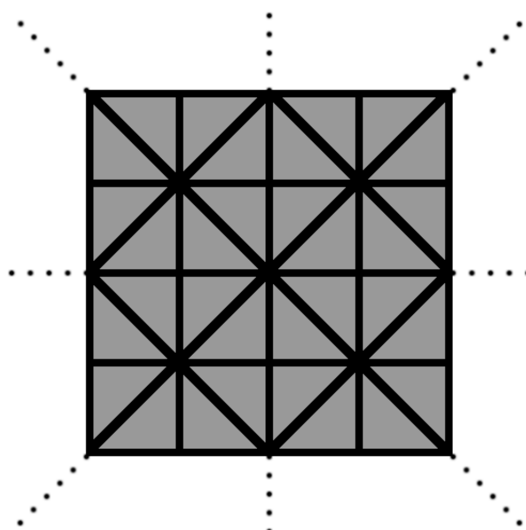

Figure 13: A tiling of the plane with right angle triangles

It is immediate from this construction that the group of symmetries for our tiling will be a Coxeter group, Γ, generated by our reflections, with relations given by the angles at which our hyperplanes meet (in the case of $\mathbb{R}^2$ the hyperplanes are straight lines).

We start by observing that $\mathbb{G}^n/\Gamma \cong \mathcal{P}$; this quotient is not of much use to us.

However, it should be obvious to see that Γ will have both finite order elements which fix points when acting on the space (i.e. reflections and rotations) as well as infinite order elements that have no fixed points. We want to isolate a finite index torsion free subgroup, $\hat{\Gamma} \subset \Gamma$. We can then quotient out by the action of this subgroup to obtain a covering of a closed manifold.

The technique we will use to do this is as follows. First, we design a homomorphism $\phi : \Gamma \to F$ where F is some finite group, with the condition that if $s \in \Gamma$ is torsion of order n, then $\phi(s) \in F$ is *also* torsion of order n (i.e. our homomorphism preserves the order of torsion elements).

It should be clear to see that by design the subgroup $K := \ker \phi \subset \Gamma$ consists only of 'translation' elements (elements of infinite order).

We could take this as our subgroup, as it will evidently act freely on our geometry. However, there is no guarantee that this will give us a 'nice' manifold – in particular, we want to force some 3-manifold to bound our 4-manifold.

So, let us again consider our tiling of $\mathbb{G}^n$. Choose a face of our tiling polytope, and let r denote the reflection across the hyperplane supported by that face. Then we will define our subgroup of Γ as

$$\hat{\Gamma} := rKr \cap K$$

This is obviously another torsion free subgroup. Thus, we can construct

$$W = \mathbb{G}^n/\hat{\Gamma}$$

Then it should be clear that $r : W \to W$ fixes some hypersurface $M \subset W$. By splitting W along M (see Figure 14) we can thus construct a 4-manifold with polytopal structure that is bounded by our 3-manifold M.

One thing we do need to pay attention to is the connectedness of M. Ideally, we want M to be connected – otherwise we can wind up with a situation where our final 4-manifold has multiple boundary components. See Figure 15 for a visualisation.

5.4 Example: Building a torus from a triangular tiling of the plane

Let us take the tiling given in Figure 12 as an example. It should be clear that the Coxeter group for this tiling is given by

$$\Gamma = \langle r_1, r_2, r_3 \,|\, r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^3 = (r_2r_3)^3 = (r_1r_3)^3 = 1\rangle$$

We will construct a homomorphism from Γ to the symmetric group of order three, with presentation

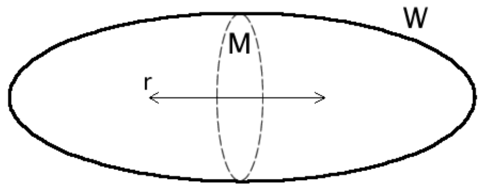

Figure 14: W^{n+1} split along M^n

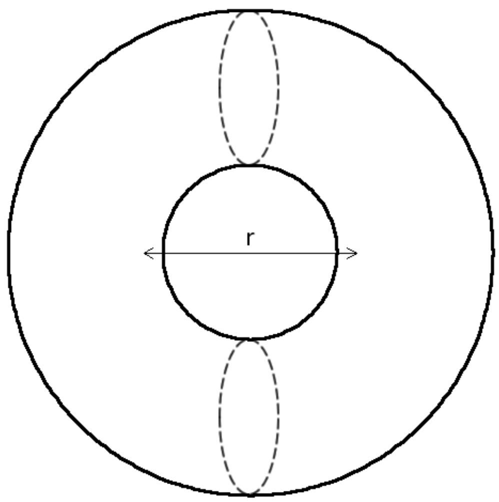

Figure 15: W^{n+1} split along a disconnected M^n

$$S_3 = \langle x, y \,|\, x^2 = y^3 = xyxy = 1 \rangle$$

Note that we can translate this into cycle notation using (for instance) $x \leftrightarrow (1\,2)$ and $y \leftrightarrow (1\,2\,3)$.

We define a homomorphism $\phi : \Gamma \to S_3$ by

$$\begin{array}{rcl} r_1 & \longmapsto & x \\ r_2 & \longmapsto & yx \\ r_3 & \longmapsto & y^2x \end{array}$$

It is easy enough to see from this that $\phi(r_1)$, $\phi(r_2)$ and $\phi(r_3)$ are all of order two (note that $y^2x = xy$), and that $\phi(r_1r_2)$, $\phi(r_2r_3)$ and $\phi(r_1r_3)$ are all of order three. Thus, our homomorphism preserves the order of the torsion elements in our 'generating' triangle.

We use a geometric argument to show that the order of every torsion element in Γ is preserved by ϕ. Some background is assumed here regarding the classification of rigid motions in the Euclidean plane; all details can be found in chapter 5 of Artin [6].

First, note that any torsion element in Γ must have a fixed point in our tiling, and is thus either a reflection or a rotation. But any reflection or rotation of our tiling will be conjugate to a reflection or rotation in our 'generating' triangle via a translation or glide reflection. That is to say that for some reflection R (rotation O) and infinite order element T (T') we have that

$$R = Tr_iT^{-1} \qquad (O = T'r_ir_jT'^{-1})$$

Since conjugation is an order preserving operation, it immediately follows that all of the torsion elements in Γ have their order preserved by ϕ. Thus, the kernel of ϕ is a subgroup containing only infinite order elements, which will act freely on our tiling of the plane.

We set $\hat{\Gamma} := \ker\phi$. Then the quotient space $\mathbb{R}^2/\hat{\Gamma}$ consists of the cosets

$$\begin{array}{ccc} \hat{\Gamma} & y\hat{\Gamma} & y^2\hat{\Gamma} \\ x\hat{\Gamma} & yx\hat{\Gamma} & y^2x\hat{\Gamma} \end{array}$$

which we can visualize by choosing representative triangles in our tiling (and ignoring triangles that are identified under infinite order actions). Obviously we need six triangles – we can choose them as per Figure 16, with side identifications obvious from the translates of the given triangles.

Thus, whichever 2-manifold corresponds to the word $abca^{-1}b^{-1}c^{-1}$ is homeomorphic to $\mathbb{R}^2/\hat{\Gamma}$. The following sequence of algebraic manipulations corresponds to repeated cutting and rejoining of the hexagon:

$$\begin{aligned} abca^{-1}b^{-1}c^{-1} &= abd \cup d^{-1}ca^{-1}b^{-1}c^{-1} \\ &= dab \cup b^{-1}c^{-1}d^{-1}ca^{-1} \\ &= dabb^{-1}c^{-1}d^{-1}ca^{-1} \\ &= ac^{-1}d^{-1}ca^{-1}d \\ &= ac^{-1}d^{-1}e^{-1} \cup eca^{-1}d \\ &= c^{-1}d^{-1}e^{-1}a \cup a^{-1}dec \\ &= c^{-1}d^{-1}e^{-1}aa^{-1}dec \\ &= decc^{-1}d^{-1}e^{-1} \\ &= ded^{-1}e^{-1} \end{aligned}$$

This word represents the planar model for the torus. Thus, we conclude that

$$\mathbb{R}^2/\hat{\Gamma} \cong T^2.$$

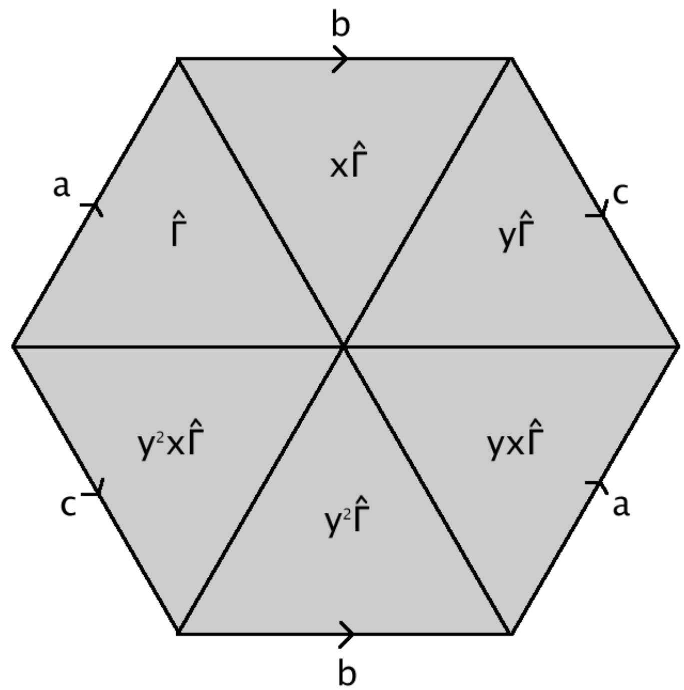

Figure 16: A graphical representation of $\mathbb{R}^2/\hat{\Gamma}$

5.5 Davis Hyperbolic 4-manifold

In the paper *A Hyperbolic 4-Manifold* [7], Michael Davis constructs a more complicated manifold using Coxeter groups (see Figure 17 for the specific groups), in a way that would be highly useful to examine when it comes to the construction of cubed 4-manifolds from cubed 3-manifolds. I will attempt to give an overview here.

Davis uses the groups G_n to construct polytopes X^n (with symmetries determined by the Coxeter groups) that tesselate various geometries: X^i for $1 \leq i \leq 3$ tesselate the corresponding S^i, and X^4 tesselates $\mathbb{H}^4$.

Davis then defines a torsion free subgroup of G_4 as follows: For each face $D \in \mathcal{D}(X^4)$, where $\mathcal{D}(X^4)$ is the set of 3-dimensional faces of X^4, a transformation t_D is constructed using the reflective symmetries of X^4. Specifically, $t_D = r_D s_D$, where r_D is the reflection of $\mathbb{H}^4$ across the hyperplane supported by D, and s_D is the reflection of $\mathbb{H}^4$ across the hyperplane through the centre of X^4 which is orthogonal to the geodesic ray from the centre of X^4 to the centre of D.

It isn't too hard to see that t_D can be thought of as a transformation; more importantly, however, is that it is of non-finite order. Davis thus defines a

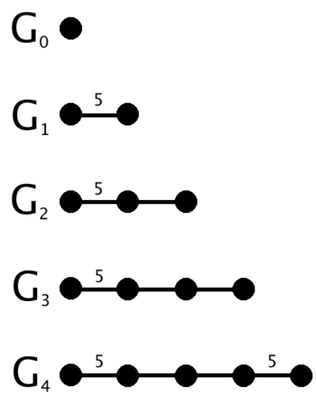

Figure 17: The Coxeter groups used in the Davis construction [7]

subgroup $K \subset G_4$ generated by the family $(t_D)_{D\in\mathcal{D}(X^4)}$. This is a finite index subgroup with acts freely on $\mathbb{H}^4$, thus the space $M^4 = \mathbb{H}^4/K$ is a hyperbolic 4-manifold.

More details about this particular manifold can be found in [7]; particular note should be taken of the geometric description of M^4. It can be shown that X^4 is a fundamental domain for K, and consequently that M^4 is obtained by identifying D with $-D$ via s_D for each $D \in \mathcal{D}(X^4)$. This gluing suggests that a cubing of non-positive curvature could be inflicted upon M^4, as well as upon its face structure.

5.6 Two final questions

All this leaves us with two final questions to consider (the answers for which sadly fall outside the time allotted to this particular project). The first question draws on Andreev's Theorem, and its relationship to non-positive cubings.

Open Question. *If we take a non-obtuse polyhedron (as given by Andreev's Theorem), when will it occur as the face of some* 4*-dimensional polytope?*

Answering this question would give us a potential 'in' to studying which cubed 3-manifolds bound cubed 4-manifolds.

The second question is suggested by the Davis construction. Remember that inside a reflection group for $\mathbb{H}^4$ Davis identifies a group that is *almost* the reflection group of a regular dodecahedron. We are interested in a kind of converse to that procedure.

Open Question. *Suppose* Γ *is the reflection group for some polyhedron. When can we find or construct a symmetry group* Λ *for some* 4*-dimensional polytope such that* Γ *is a subgroup of* Λ*?*

Answering this question would obviously provide a great deal of information that (in conjunction with the aforementioned construction) could allow us to answer the more general question posed back at the start.

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