Integrable Systems Related to the Two-Dimensional Euler Fluid Flow on a Rotating Sphere

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1. Introduction

Euler fluid flow refers to incompressible, inviscid fluid flow. On the surface of a rotating unit sphere, this gives rise to the Barotropic Vorticity Equation (BVE) (Lynch, 2009).

$$\frac{\partial q}{\partial t} + J[\Delta_s^{-1}(q - 2\Omega\cos\theta), q] = 0$$

Where Δ_s^{-1} is the inverse spherical Laplacian, J is the spherical Jacobian, θ the co-latitude, ϕ the longitude, and q the vorticity of the fluid, defined as the curl of the vector field. The vorticity hence corresponds to the rotation of the fluid.

The typical method for computing solutions to the BVE is to decompose the vorticity into spherical harmonics.

$$q = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_{lm}(t) Y_{lm}(\theta, \phi)$$

Where the $Y_{lm}(\theta, \phi)$ are the Laplacian spherical harmonic functions, and the modes $q_{lm}(t)$ can be found by solving the ordinary differential equations given by the Lie-Poisson structure (Zeitlin, 2004).

$$\frac{dq_{lm}}{dt} = \sum_{l'm'} \sum_{l''m''} \frac{\gamma_{lm\,l'm'}^{l''m''}}{l'(l'+1)} (-1)^{m'} \left(q_{l'-m'} - 4\Omega\sqrt{\frac{\pi}{3}}\,\delta(\{l'-1\})\delta(\{m'\}) \right) q_{l''m''}$$

Where $\sum_{lm}$ is understood to mean $\sum_{l=1}^{\infty}\sum_{m=-l}^{l}$, δ is the Dirac Delta function and $\gamma_{lm\,l'm'}^{l''m''}$ are the structure coefficients. Note that we ignore the $l = 0$ case from here on as the differential equation for q_{00} is always trivial.

To approximate the solution to these ODEs, truncation is performed. Naïve truncation comes with the cost of breaking the Poisson structure. Zeitlin (2004) gives a formula for $f^{(N)\,l''m''}_{lm\,l'm'}$ which are to be used as the structure coefficients when truncating the system at $l = N$ which preserves the Poisson structure. The new structure coefficients are dependent on the degree of truncation, and as $n \to \infty$, $f^{(N)\,l''m''}_{lm\,l'm'} \to \gamma^{l''m''}_{lm\,l'm'}$.

The Poisson structure allows us to express the time evolution of any function, say g, of the q_{lm} by taking the Poisson bracket of g with H, the Hamiltonian.

$$\frac{dg}{dt} = \{g, H\} = \sum_{lm} \sum_{l'm'} \sum_{l''m''} f^{(N)\,l''m''}_{lm\,l'm'} \, q_{l''m''} \frac{dg}{dq_{lm}} \frac{dH}{dq_{l'm'}} = \nabla g^T \, P(q_{lm}) \, \nabla H$$

Where $P(q_{lm})$ is a matrix with the $(i,j)^{th}$ element given by $\sum_{l''m''} f^{(N)\,l''m''}_{r(i)\,r(j)} \, q_{l''m''}$ and $r(i)$ is a bijective map $\{i : i \in \mathbb{N}, 1 \le i \le (N+1)^2 - 1\} \longmapsto \{(l,m) : 1 \le l \le N, -l \le m \le l\}$. Essentially, $r(i)$ gives an ordering to the q_{lm}.

The Hamiltonian is the kinetic energy of the fluid, and is given by the following integral over the unit sphere S^2 (Zeitlin, 2004).

$$H = -\frac{1}{2} \int_{S^2} (q + 2\Omega \cos\theta) \, \Delta_s^{-1} q$$

To put the Hamiltonian in an appropriate form for use in calculating the Poisson bracket, we substitute $q = \sum_{lm} q_{lm} Y_{lm}$ and simplify using the conjugation property $Y_{lm} = (-1)^m \overline{Y_{l-m}}$ and the orthogonality relations $\int_{S^2} Y_{lm} \overline{Y_{l'm'}} = \delta_{ll'} \delta_{mm'}$ where $\delta_{ij} = 1 \; if \; i = j$ and $\delta_{ij} = 0 \; if \; i \neq j$. The result is given by:

$$H = 2\Omega \sqrt{\frac{\pi}{3}} q_{10} + \frac{1}{2} \sum_{lm} \frac{q_{lm} q_{l-m} (-1)^m}{l(l+1)}$$

Our Poisson system can be thought of as a generalisation of a canonical Hamiltonian system. Whereas the Poisson bracket of two functions in the canonical framework is given by $\{f, g\} = \nabla f^T \, J \, \nabla g$ where the matrix J is symplectic and constant, in this non-canonical framework we have $\{f, g\} = \nabla g^T \, P(q_{lm}) \, \nabla g$ where the matrix $P(q_{lm})$ is anti-symmetric and linear in the modes q_{lm}.

2. Integrabilitiy N=2

For N=2, excluding the trivial q_{00} mode, there are 8 modes (q_{11}, q_{21}, q_{22}, q_{1-1}, q_{2-1}, q_{2-2}, q_{10}, q_{20}). The rank of the Poisson structure matrix $P(q_{lm})$ is 6. This degeneracy leads to conserved quantities known as Casimirs, which have the interesting property that the Poisson bracket of a Casimir with any arbitrary function (as opposed to just with the Hamiltonian) is zero. The method for finding the Casimirs is given by Zeitlin (2004), and they turn out to be of homogenous degree 2 and 3.

$$P(q_{lm}) = \mathrm{i}\sqrt{\frac{\pi}{3}} \times$$
$$\begin{pmatrix} 0 & \sqrt{2}q_{22} & 0 & q_{10} & \sqrt{3}q_{20} & \sqrt{2}q_{2-1} & q_{11} & \sqrt{3}q_{21} \\ -\sqrt{2}q_{22} & 0 & 0 & \sqrt{3}q_{20} & q_{10} & -\sqrt{2}q_{1-1} & q_{21} & \sqrt{3}q_{11} \\ 0 & 0 & 0 & \sqrt{2}q_{21} & -\sqrt{2}q_{11} & -2q_{10} & 2q_{22} & 0 \\ -q_{10} & -\sqrt{3}q_{20} & -\sqrt{2}q_{21} & 0 & -\sqrt{2}q_{2-2} & 0 & -q_{1-1} & -\sqrt{3}q_{2-1} \\ -\sqrt{3}q_{20} & -q_{10} & \sqrt{2}q_{11} & \sqrt{2}q_{2-2} & 0 & 0 & -q_{2-1} & -\sqrt{3}q_{1-1} \\ -\sqrt{2}q_{2-1} & \sqrt{2}q_{1-1} & 2q_{10} & 0 & 0 & 0 & -2q_{2-2} & 0 \\ -q_{11} & -q_{21} & -2q_{22} & q_{1-1} & q_{2-1} & 2q_{2-2} & 0 & 0 \\ -\sqrt{3}q_{21} & -\sqrt{3}q_{11} & 0 & \sqrt{3}q_{2-1} & \sqrt{3}q_{1-1} & 0 & 0 & 0 \end{pmatrix}$$

For the system to be integrable, we require a number of *integrals*, or conserved quantities, equal to half the degrees of freedom in the system. Three such integrals were found by looking for polynomials g of the q_{lm} such that $dg/dt = \nabla g^T \, P(q_{lm}) \nabla H = 0$. These integrals are:

$$I_1 = q_{10}$$
$$I_2 = q_{11}q_{1-1}$$
$$\mathrm{I}_3 = 3\sqrt{2}q_{11}^2 q_{2-2} - 6q_{10}q_{11}q_{2-1} + 2\sqrt{3}q_{10}^2 q_{20}$$
$$+ 2\sqrt{3}q_{1-1}q_{11}q_{20} - 6q_{1-1}q_{10}q_{21} + 3\sqrt{2}q_{1-1}^2 q_{22}$$

Note that the Hamiltonian corresponds to the energy of the system and is also a conserved quantity, but it turns out not to be independent of the other integrals; it can be expressed as a function of the degree 2 Casimir, I_1 and I_2.

3. Exact Solution N=2

The system for $N = 2$ is exactly solvable. The differential equations are given by $\frac{dq_{lm}}{dt} = (\nabla q_{lm})^T P(q_{lm}) \, \nabla H$. The system separates into two subsystems, the first of which is:

$$\frac{d}{dt}\begin{pmatrix} q_{10} \\ q_{11} \\ q_{1-1} \end{pmatrix} = \begin{pmatrix} 0 \\ -i\Omega\, q_{11} \\ i\Omega\, q_{1-1} \end{pmatrix}$$

Using the fact that in order to have real solutions requires $q_{lm} = (-1)^m q_{l-m}$, the solution to the above equation is just the harmonic oscillator in complex variables where α, A are constant:

$$\begin{pmatrix} q_{10} \\ q_{11} \\ q_{1-1} \end{pmatrix} = \begin{pmatrix} \alpha \\ A\, e^{-i\Omega t} \\ -A\, e^{i\Omega t} \end{pmatrix}$$

The second subsystem can be expressed in the form where L is a 5x5 matrix linear in $q_{10},\ q_{21}, q_{22}$:

$$\frac{d}{dt}\begin{pmatrix} q_{20} \\ q_{21} \\ q_{2-1} \\ q_{22} \\ q_{2-2} \end{pmatrix} = L(q_{10}, q_{11}, q_{1-1}) \begin{pmatrix} q_{20} \\ q_{21} \\ q_{2-1} \\ q_{22} \\ q_{2-2} \end{pmatrix}$$

Since $q_{lm} = (-1)^m q_{l-m}$, the real and imaginary parts of each q_{lm} and q_{l-m} are the same up to a factor of -1. This can be used to transform the entire 8 q_{lm} variable system into a system of 8 purely real variables which correspond to the 8 unique real and imaginary parts of the q_{lm}.

In real variables, substituting in the exact solution for the first subsystem, the second subsystem is then of the form $dx/dt = C(t)x$, where $C(t)$ is $2\pi/\Omega$ periodic. Floquet theory tells us there is a transformation, $y = S(t)x$ such that $dy/dt = Dy$ where D is constant. Finding this transformation requires finding S such that $D = \dot{S}S^{-1} + SCS^{-1}$ is constant. Assume that S is orthogonal, so $D = \dot{S}S^T + SCS^T$. Try the substitution $S = e^{KT}$, K constant, and differentiate with respect to time, so $0 = \dot{C} + CK - KC$. Solving for K shows that there is indeed a solution, and we have the required transformation.

There is then an orthogonal transformation $z = Ty$, found by taking the rows of T to be the real and imaginary parts of the eigenvectors of D. The equation for z is then very simple:

$$\frac{dz}{dt} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & -2\omega & 0 \end{pmatrix} z$$

Where $\omega = \sqrt{\frac{A^2+\alpha^2}{12\pi}}$.

The solutions, where β, B_1, B_2 are constant, are given by:

$$z_1 = \beta \qquad z_2 = B_1 \sin \omega t \qquad z_4 = B_2 \sin 2\omega t$$
$$z_3 = B_1 \cos \omega t \qquad z_5 = B_2 \cos 2\omega t$$

The explicit equations for the $q_{lm}(t)$ can be found by applying the inverse transformations to these solution.

Interestingly, as the frequencies are always rational multiples of each-other regardless of parameter choice, there is essentially only one frequency describing the second subsystem. Since there is only one other frequency in the entire system, namely Ω from the harmonic oscillator, we have two degrees of freedom in our 8 variable system. Hence we would expect to find exactly six independent conserved quantities. At this stage we have 5 conserved quantities; the two Casimirs, I_1, I_2 and I_3. Noting that:

$$\left(\frac{dz_2}{dt}\right)^2 + \left(\frac{dz_3}{dt}\right)^2 = z_3^2 + z_2^2 = B_1$$

Which is independent of time. Applying the inverse transformations to this expression gives an expression of homogenous degree six which is indeed conserved and independent of the other integrals. However, the drawback of this method is that we expect the new integral, I_4, could be expressed more succinctly; the expression derived from the above method can likely be expressed as a polynomial function of I_1, I_2, I_3, I_4, C_1 and C_2.

Of note is that by setting $q_{10} = 4\sqrt{\pi/3}\,\Omega$ and all but one mode q_{lm} to zero, we get:

$$\frac{dq_{lm}}{dt} = -\frac{2\Omega m}{l(l+1)} q_{lm}$$

The solutions to these are the Rossby-Haurwitz waves, which are well known solutions the linearised Barotropic Vorticity Equation (Longuet-Higgins, 1964).

4. Canonical Form N=2

To attempt to express the system in action/angle form, the system can be transformed into canonical variables. Because of the degeneracy in the Poisson structure (we have 8 variables but the rank of the Poisson structure matrix is 6), the system will have 6 variables in canonical form. There exists a method for the transformation to canonical variables (Zeitlin, 1991).

The canonical variables obey the relations:

$$\dot{r_i} = \frac{\partial s_i}{\partial H}, \qquad \dot{s_i} = -\frac{\partial r_i}{\partial H}$$

Upon transformation to canonical variables, many of our conserved quantities are no longer independent. We are left with 3 independent integrals, but the fact that we have 6 variables and we know the exact solution in the q_{lm} has 2 degrees of freedom would suggest that four integrals exist; no new integral could be found by using brute forces searches up to degree 10 polynomials. The integrals are:

$$\lambda_1 = r_3 s_3 - r_1 s_1$$
$$\lambda_2 = (r_1 s_2 + r_2 s_3)(r_2 s_1 + r_3 s_2)$$
$$\lambda_3 = r_1 s_1 + r_2 s_2 + r_3 s_3$$

The Hamiltonian can be expressed in the following form, though for calculation purposes it should be expressed in terms of the r_i and s_i.

$$H = \frac{\Omega}{\sqrt{2}}\lambda_1 + \frac{1}{12}\lambda_1^2 + \frac{1}{6}\lambda_2 + \frac{1}{18}\lambda_3^2$$

Further investigation could focus on finding action/angle variables for this Hamiltonian using Hamilton-Jacobi theory. Knowing this transformation would aid in describing all possible motions of the $q_{lm}(t)$.

5. Notes on N=3

Truncation at N=3 yields a system with similar structure; the vector field separates into 3 sun-subsystems. Denote $Q_i = (q_{i-i}, q_{i(-i+1)}, \dots, q_{i(i-1)}, q_{ii})$ where $i = 1,2,3$. The differential equations can then be written:

$$\frac{dQ_1}{dt} = CQ_1$$

$$\frac{dQ_3}{dt} = L(Q_1)\, Q_3$$

$$\frac{dQ_2}{dt} = \hat{L}(Q_1, Q_3)\, Q_2$$

Where C is constant, and $L, \hat{L}$ are linear in the q_{lm}. The solution to the Q_1 subsystem is again just the harmonic oscillator; the Q_3 subsystem can be solved using Floquet theory giving time periodic solutions, so Floquet theory can be applied again to the Q_2 system, although finding the required transformation would be perhaps prohibitively computationally intensive.

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