The Classical groups

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Lie Theory was developed at the end of the 19th century by Sophus Lie. He originally wanted to construct the theory behind continuous (linear) groups, that he hoped would unify the study of differential equations. His work made several important breakthroughs in the study of geometry. Lie Groups are now an essential algebraic framework used in the study of differential geometry.

Lie Theory is often a post-graduate course that contains many aspects of differential geometry. As a student having completed only second year, this project was designed as an introduction to Lie Groups with the use of linear (or matrix) groups. These groups are simpler to visualise and can be studied with more initiative, geometric ideas. In particular, the classical groups are certain linear groups that include the rotation, unitary and orthogonal groups.

My project involved examining the structures of these groups. The exponential map locally associates the group with its tangent space at the identity (or Lie Algebra). As this tangent space is a vector space, one can exploit the tangent's linear structure to move within the group using the map. This map is essential to develop coordinates for manifolds in later study.

Throughout my project, I have been working on several examples, including how the homeomorphic property of the exponential map fails when working with non-closed groups, for example, the irrational line on the torus. Here, the inverse map is not surjective due to the denseness of the lines.

While Lie Algebras are useful, O(3), the 3×3 orthogonal group, and SO(3), the 3×3 special orthogonal group, are two groups which, although different (SO(3) is a strict subgroup of O(3)) they have the same Lie Algebra. Hence, one cannot ignore the Lie Group structure in favour of the Lie Algebra. Another example of this is SO(3) and SU(2), the 2×2 special unitary matrices, whose Lie algebras are also isomorphic. This isomorphism can be found using either the unit quaterions or with the adjoint representation of SO(3), and leads into another broad area of study, Representation Theory.

Finally I have looked into the relationship between linear groups and Lie Algebras and where isomorphisms occur. When two linear groups are isomorphic, the differential of the isomorphism at the identity (the Lie Map) is an isomorphism of the Lie Algebras. When two Lie Algebras are isomorphic, under certain conditions, the isomorphism will lift to a group homomorphism.

This project has taught me invaluable skills. I have learnt to use the program LaTeX (including Beamer) and have experienced firsthand how research is carried out in a mathematics department. It has allowed me to talk to various people (including at the Big Day In) and garner an interest in further study in mathematics. As part of the project, I also wrote a report that details the topics I have studied, designed to be a miniature honours thesis.

This project allowed me to use the knowledge from second year courses, Multivariable and Complex Calculus, Differential Equations, Algebra and Real Analysis. It cemented various ideas from these subjects, including the uniqueness of solutions to differential equations and the Inverse and Implicit function theorems. The mathematics I have learnt within the project will be of great benefit to me; it has helped prepare me for my mathematics career as it leads into Lie Theory and Differential Geometry.

I am currently hoping to continue to a PhD within differential geometry, and highly recommend an AMSI Summer Research Scholarship to all mathematics students. Thank-you to AMSI, for the scholarship, and to my supervisor, Dr Thomas Leistner, for his mentorship throughout the program.

References:

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