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Symbolic Dynamics

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The project extends the link between symbolic dynamics and directed graphs into two dimensions. We investigated a method for converting certain shift-spaces into 2-dimensional graphs called *2-graphs.* We then looked for conditions on the shift-space which produce *aperiodic* 2-graphs.

Our shift spaces consist of infinite lattices called *2-spaces* of 0s and 1s. The allowed lattices are those for which a given "domino" placed anywhere in the lattice has entries which satisfy some polynomial rule. For example, the "sock" domino corresponds to the polynomial $1 + x + y$ and sock-shaped dominos in an allowed 2-space must have one of four configurations:

$$\begin{matrix} 0 & \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & \\ 1 & 1 \end{matrix}.$$

Fort a chosen domino/polynomial, we want to form the skeleton of the corresponding 2-graph. The vertices are the allowed dominos. The component graphs can be formed by determining vertex adjacency matrices for edges in the x-direction and y-direction. The x-edges correspond to shifting the domino horizontally in the 2-space and y-edges correspond to shifting it vertically.

Not all choices of domino/polynomial will generate the skeleton of a 2-graph. We must check that the Factorisation Property holds: between every pair of vertices, there exists an x - y path if and only if there exists a y - x path. We can check that the property holds by verifying that the adjacent matrices commute. We discovered that the dominos which do work are staircase shaped.

Every infinite path of alternating degrees in the skeleton generates an element of the shift space. We are interested in identifying aperiodic cycles in the skeletons by observing patterns in the 2-spaces. Our strategy is to form the aperiodic path

$$\beta\alpha\beta\alpha\alpha\beta\alpha\alpha\alpha \ldots$$

where β is a cycle of the skeleton and α is a loop of the source of β. Taking α as a loop at the zero vertex helps us recognise patterns easily. For example, we discovered that the "square" domino which corresponds to the polynomial $1 + x + y + xy = 0 \pmod 2$ with the cycle given by

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$$\beta = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \rightarrow \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \rightarrow \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} \rightarrow \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$$

$$\alpha = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \rightarrow \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \rightarrow \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$$

gives a clearly aperiodic pattern in the 2-space; a checkerboard of increasing rectangular blacks of 0s and 1s. If we consider individual rows and columns, the pattern of 0s and 1s never repeats, confirming aperiodicity.