

On the existence of orthonormal geodesic bases for Lie algebras

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Abstract. We show that every unimodular Lie algebra, of dimension at most 4, equipped with an inner product, possesses an orthonormal basis comprised of geodesic elements. On the other hand, we give an example of a solvable unimodular Lie algebra of dimension 5 that has no orthonormal geodesic basis, for any inner product.

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1 Introduction

Let \mathfrak{g} be a Lie algebra equipped with an inner product $\langle \cdot, \cdot \rangle$. Consider the corresponding simply connected Lie group G equipped with the left invariant Riemannian metric determined by $\langle \cdot, \cdot \rangle$. A nonzero element $Y \in \mathfrak{g}$ is said to be a *geodesic vector* if the corresponding left invariant vector field on G is a geodesic vector field. In terms of the Levi-Civita connection ∇ , this means that $\nabla_Y Y = 0$. This has a simple equivalent expression in terms of the Lie bracket [8, 9, 11, 2], which we state as a definition.

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Definition 1. Let \mathfrak{g} be a Lie algebra equipped with an inner product $\langle \cdot, \cdot \rangle$. Then a nonzero element $Y \in \mathfrak{g}$ is a *geodesic vector* if $[X, Y]$ is perpendicular to Y for all $X \in \mathfrak{g}$.

Note that some authors use the term *homogeneous geodesic*, to distinguish them from general geodesics on the underlying Lie group. Some authors insist further that Y have unit length. For the more general case of totally geodesic subalgebras of Lie algebras, see [11, 2, 3].

Remark 1. A useful equivalent reformulation of the definition of a geodesic vector is as follows: $Y \in \mathfrak{g}$ is geodesic if and only if $\text{Im}(\text{ad}(Y))$ is contained in the orthogonal complement of Y .

Every Lie algebra possesses at least one geodesic vector [7, 9, 2]. In [9] it is shown that semisimple Lie algebras possess an orthonormal basis comprised of geodesics vectors, for every inner product. Results for certain solvable algebras are given in [4]. In the present paper we examine the existence of geodesic bases in algebras of low dimension.

2 Preliminary observations

First notice that if a Lie algebra \mathfrak{g} , equipped with an orthonormal inner product, has a basis $\{X_1, \dots, X_n\}$ of geodesic vectors, then for all i, j , the element $[X_i, X_j]$ is orthogonal to both X_i and X_j . Consequently, the adjoint maps $\text{ad}(X_i)$ have zero trace, and hence \mathfrak{g} is unimodular. So the natural problem is to determine which unimodular Lie algebras possess an orthonormal basis comprised of geodesic vectors, for some inner product. We begin with two elementary observations.

Proposition 1. *Let \mathfrak{g} be a nilpotent Lie algebra equipped with an inner product $\langle \cdot, \cdot \rangle$. Then there is an orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ comprised of geodesic vectors.*

PROOF. The proof is by induction on the dimension of \mathfrak{g} . The proposition is trivial when $\dim(\mathfrak{g}) = 1$. Suppose that $\dim(\mathfrak{g}) = n + 1$. Since \mathfrak{g} is nilpotent, its centre is nonzero. Let Z be a central element of length one and set $\mathfrak{z} := \text{Span}(Z)$. Let $\pi : \mathfrak{g} \rightarrow \mathfrak{h} := \mathfrak{g}/\mathfrak{z}$ be the natural quotient map and give \mathfrak{h} the inner product for which the restriction of π to the orthogonal complement \mathfrak{z}^\perp of \mathfrak{z} is an isometry. By the inductive hypothesis, \mathfrak{h} possesses an orthonormal basis $\{X_1, \dots, X_n\}$ comprised of geodesic vectors. So, by definition, for each pair i, j , the element $[X_i, X_j]$ is perpendicular to both X_i and X_j . For each i , let \bar{X}_i denote the unique element of \mathfrak{z}^\perp with $\pi(\bar{X}_i) = X_i$. So for each pair i, j , the element $[\bar{X}_i, \bar{X}_j]$ is perpendicular to both \bar{X}_i and \bar{X}_j . Thus, as Z is central, the elements \bar{X}_i are geodesic vectors. Furthermore, as Z is central, Z is also geodesic. Hence

$\{\bar{X}_1, \dots, \bar{X}_n, Z\}$ is an orthonormal basis of geodesic vectors. \square

Proposition 2. *Let \mathfrak{g} be a unimodular Lie algebra having a codimension one abelian ideal \mathfrak{h} . Then, for every inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} , there is an orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ comprised of geodesic vectors.*

PROOF. We make use of the following linear algebra result.

Lemma 1 ([14, Theorem 10]). *Suppose that A is a real square matrix with zero trace. Then there is an orthogonal matrix Q such that QAQ^{-1} has zero diagonal.*

Let $X \in \mathfrak{g}$ be a unit vector orthogonal to \mathfrak{h} . Note that X is geodesic since \mathfrak{h} contains the derived algebra \mathfrak{g}' of \mathfrak{g} . Choose an orthonormal basis for \mathfrak{h} and let A denote the matrix representation of the restriction $\text{ad}_{\mathfrak{h}}(X)$ to \mathfrak{h} of the adjoint map $\text{ad}(X)$. Since \mathfrak{g} is unimodular, the matrix A has zero trace. By Lemma 1, there is an orthonormal basis $\{Y_1, Y_2, \dots, Y_n\}$ for \mathfrak{h} relative to which $\text{ad}_{\mathfrak{h}}(X)$ has zero diagonal. Consequently for each i , the element $[X, Y_i]$ is perpendicular to Y_i . Thus since \mathfrak{h} is abelian, the elements Y_i are geodesic vectors. Hence $\{X, Y_1, Y_2, \dots, Y_n\}$ is a geodesic basis for \mathfrak{g} . \square

3 Main Results

It is well known that in dimension less than or equal to three, there are only 5 nonabelian unimodular real Lie algebras; see [16]. Milnor's classification [10] proceeds by considering, for an orthonormal basis $\{X_1, X_2, X_3\}$, the linear map $L(X_i \times X_j) := [X_i, X_j]$, where \times denotes the vector cross product. The matrix of L relative to $\{X_1, X_2, X_3\}$ is symmetric and so its eigenvectors form an orthonormal basis $\{Y_1, Y_2, Y_3\}$ with, by construction,

$$\begin{aligned} [Y_2, Y_3] &= \lambda_1 Y_1, \\ [Y_3, Y_1] &= \lambda_2 Y_2, \\ [Y_1, Y_2] &= \lambda_3 Y_3, \end{aligned}$$

for real coefficients λ_i . So the basis elements Y_1, Y_2, Y_3 are geodesic vectors.

In dimension 4, there are infinitely many isomorphism classes of unimodular real Lie algebras; see [12, 5, 15]. Instead of using the classification, we will argue directly. The following result resolves a question raised in [13].

Theorem 1. *Let \mathfrak{g} be a unimodular Lie algebra of dimension 4 equipped with an inner product $\langle \cdot, \cdot \rangle$. Then there is an orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ comprised of geodesic vectors.*

PROOF. Let \mathfrak{g} be as in the statement of the theorem. First observe that if \mathfrak{g} is not solvable, then from Levi's Theorem and the classification of semisimple

algebras [6], there is a Lie algebra isomorphism $\mathfrak{g} \cong \mathfrak{h} \oplus \mathbb{R}$ where \mathfrak{h} is a simple Lie algebra of dimension three; so \mathfrak{h} is isomorphic to either $\mathfrak{so}(3, \mathbb{R})$ or $\mathfrak{sl}(2, \mathbb{R})$ though we won't need this fact. Let $W \in \mathfrak{g}$ be a unit vector orthogonal to \mathfrak{h} . Since $\mathfrak{h} = \mathfrak{g}'$, the element W is geodesic. Note that W can be written uniquely in the form $W = X + Z$, where Z is in the centre of \mathfrak{g} and $X \in \mathfrak{h}$. As we discussed above, \mathfrak{h} has an orthonormal basis $\{Y_1, Y_2, Y_3\}$ of elements that are geodesic vectors of \mathfrak{h} . Then for each $i = 1, 2, 3$,

$$[W, Y_i] = [X, Y_i] \in \text{Im}(\text{ad}(Y_i)|_{\mathfrak{h}}),$$

which is perpendicular to Y_i by Remark 1. Hence Y_i is geodesic in \mathfrak{g} and thus $\{Y_1, Y_2, Y_3, W\}$ is an orthonormal basis of geodesic vectors. So we may assume that \mathfrak{g} is solvable.

Consider the derived algebra $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$. As \mathfrak{g} is solvable, \mathfrak{g}' is nilpotent [1, Chap. 1.5.3]. If $\dim(\mathfrak{g}') = 0$, the algebra \mathfrak{g} is abelian and the theorem holds trivially. We consider three cases according to the remaining possibilities for the dimension of \mathfrak{g}' .

If $\dim(\mathfrak{g}') = 1$, let $\mathfrak{g}' = \text{Span}(W)$. For each $X \in \mathfrak{g}$ we have $[X, Y] \in \mathfrak{g}'$ for all $Y \in \mathfrak{g}$ and hence $[X, W] = \text{tr}(\text{ad}(X))W = 0$. Thus \mathfrak{g}' is contained in the centre of \mathfrak{g} . Consequently \mathfrak{g} is nilpotent, and the theorem holds by Proposition 1.

If $\dim(\mathfrak{g}') = 3$, then either \mathfrak{g}' is abelian or \mathfrak{g}' is isomorphic to the Heisenberg Lie algebra \mathfrak{h}_1 . If \mathfrak{g}' is abelian, the required result follows from Proposition 2. So suppose that \mathfrak{g}' is isomorphic to the Heisenberg Lie algebra. Choose an orthonormal basis $\{X, Y, Z\}$ of \mathfrak{g}' with Z in the centre of \mathfrak{g}' and $[X, Y] = \lambda Z$ for some $\lambda \neq 0$. Let $W \in \mathfrak{g}$ be a unit vector orthogonal to \mathfrak{g}' . Note that W is geodesic. Let A denote the matrix representation of the restriction $\text{ad}_{\mathfrak{g}'}(W)$ to \mathfrak{g}' of the adjoint map $\text{ad}(W)$. Since the centre of \mathfrak{g}' is a characteristic ideal [1, Chap. 1.1.3], it is left invariant by $\text{ad}(W)$. Hence A has the form

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{pmatrix},$$

for reals a, b, c, d, e, f, g . By the Jacobi identity,

$$[\text{ad}(W)X, Y] + [X, \text{ad}(W)Y] = \text{ad}(W)\lambda Z = g\lambda Z.$$

Thus

$$[aX + cY + eZ, Y] + [X, bX + dY + fZ] = (a + d)\lambda Z = g\lambda Z,$$

and so $a + d = g$. Since $\text{ad}(W)$ has zero trace, $a + d + g = 0$, and hence $a + d = g = 0$. So, by Lemma 1, we can make an orthonormal change of basis

for the space $\text{Span}(X, Y)$ so that relative to the new basis, $a = d = 0$. Then $\{W, X, Y, Z\}$ is an orthonormal geodesic basis for \mathfrak{g} .

Finally, if $\dim(\mathfrak{g}') = 2$, then \mathfrak{g}' is abelian. Let $\{X, Y\}$ be an orthonormal basis for the orthogonal complement \mathfrak{g}'^\perp to \mathfrak{g}' . Note that $[X, Y] \in \mathfrak{g}'$ and so as \mathfrak{g}' is abelian, the adjoint map $\text{ad}([X, Y])$ acts trivially on \mathfrak{g}' . Thus, since $\text{ad}(X) \circ \text{ad}(Y) - \text{ad}(Y) \circ \text{ad}(X) = \text{ad}([X, Y])$ by the Jacobi identity, we have that the restrictions $\text{ad}_{\mathfrak{g}'}(X)$ and $\text{ad}_{\mathfrak{g}'}(Y)$ commute. Since $\text{ad}_{\mathfrak{g}'}(X)$ and $\text{ad}_{\mathfrak{g}'}(Y)$ have zero trace, relative to any choice of orthonormal basis for \mathfrak{g}' , they have matrix representations in $\mathfrak{sl}(2, \mathbb{R})$. However, it is well known and easy to see that the maximum abelian subalgebras of $\mathfrak{sl}(2, \mathbb{R})$ have dimension one. Consequently, by orthonormal change of basis for \mathfrak{g}'^\perp if necessary, we may assume that $\text{ad}_{\mathfrak{g}'}(Y) \equiv 0$. Then $\text{Span}(Y, \mathfrak{g}')$ is a codimension one abelian ideal and the result follows from Proposition 2. \square

Note that in Propositions 1, 2 and Theorem 1, the required orthonormal geodesic basis is obtained for *every* inner product. The following example gives a solvable unimodular Lie algebra that has no orthonormal geodesic basis for *any* inner product.

Example 1. Consider the 5-dimensional algebra \mathfrak{g} with basis $B = \{X_1, \dots, X_5\}$ and relations

$$\begin{aligned} [X_1, X_2] &= 3X_2 & [X_2, X_3] &= X_4 \\ [X_1, X_3] &= -4X_3 & [X_2, X_4] &= X_5 \\ [X_1, X_4] &= -X_4 \\ [X_1, X_5] &= 2X_5. \end{aligned}$$

The ideal \mathfrak{n} generated by X_2, \dots, X_5 is the (unique) 4 dimensional filiform nilpotent Lie algebra. The adjoint map $\text{ad}(X_1)$ is obviously a Lie derivation of \mathfrak{n} , so the Jacobi identities of \mathfrak{g} hold. Clearly, \mathfrak{g} is solvable and unimodular, but not nilpotent. We will show that \mathfrak{g} has no orthonormal basis comprised of geodesic vectors, for any inner product. Equip \mathfrak{g} with an inner product and suppose it has a basis (not necessarily orthonormal) of geodesic elements Y_1, \dots, Y_5 . Consider an arbitrary geodesic vector $Y = a_1X_1 + \dots + a_5X_5 \in \mathfrak{g}$. Let $V_i := \text{Span}(X_i, X_{i+1}, \dots, X_5)$. Relative to the basis B , the matrix representation of $\text{ad}(Y)$ is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -3a_2 & 3a_1 & 0 & 0 & 0 \\ 4a_3 & 0 & -4a_1 & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & 0 \\ -2a_5 & -a_4 & 0 & a_2 & 2a_1 \end{pmatrix}. \quad (1)$$

A priori, there are five cases:

- (1) $Y \in V_5$; that is, $a_1 = a_2 = 0, a_3 = 0, a_4 = 0, a_5 \neq 0$,
- (2) $Y \in V_4 \setminus V_5$; that is, $a_1 = a_2 = 0, a_3 = 0, a_4 \neq 0$,
- (3) $Y \in V_3 \setminus V_4$; that is, $a_1 = a_2 = 0, a_3 \neq 0$,
- (4) $Y \in V_2 \setminus V_3$; that is, $a_1 = 0, a_2 \neq 0$,
- (5) $Y \in V_1 \setminus V_2$; that is, $a_1 \neq 0$.

In fact, case (1) is impossible as otherwise Y is a nonzero multiple of X_5 but $\text{Im}(\text{ad}(Y)) = V_5$, which must be orthogonal to Y by Remark 1. Similarly, case (2) is impossible as otherwise $Y \in V_4$ but $\text{Im}(\text{ad}(Y)) = V_4$, which has to be orthogonal to Y by Remark 1.

In case (3), $\text{Im}(\text{ad}(Y)) = \text{Span}(X_3 + \frac{a_4}{4a_3}X_4 - \frac{2a_5}{4a_3}X_5, X_4 + \frac{a_4}{a_3}X_5)$.

In case (4), $\text{Im}(\text{ad}(Y)) = \text{Span}(X_2 - \frac{4a_3}{3a_2}X_3, X_4, X_5)$ and in particular, Y is orthogonal to V_4 .

In case (5), $\text{Im}(\text{ad}(Y)) = V_2$, which is the orthogonal complement of Y by Remark 1. It follows that, up to a constant multiple, there is at most one geodesic vector Y with $a_1 \neq 0$. Thus, there is at most one basis element, Y_1 say, with $Y_1 \notin V_2$.

So we have established that

$$Y_2, Y_3, Y_4, Y_5 \in V_2 \setminus V_4.$$

But if $Y_i \in V_2 \setminus V_3$, then as we saw in case (4), Y_i is orthogonal to V_4 . So for dimension reasons, there are at most two of the Y_i in $V_2 \setminus V_3$.

Now suppose the Y_i are orthogonal. If two of the Y_i are in $V_2 \setminus V_3$, they would both be orthogonal to V_4 . This would force the remaining two Y_i to be in V_4 , which we have seen is impossible. So only one of the Y_i , say Y_2 , can be in $V_2 \setminus V_3$. So Y_3, Y_4, Y_5 are in $V_3 \setminus V_4$. Note that V_3 is left invariant by $\text{ad}(X_1)$. Relative to the orthonormal basis $\{Y_3, Y_4, Y_5\}$ for V_3 , the map $f := \text{ad}(X_1)|_{V_3}$ has zero diagonal, because the elements Y_3, Y_4, Y_5 are geodesic vectors. So f has zero trace. But relative to the basis X_3, X_4, X_5 , it is clear that f has trace $-4 - 1 + 2 = -3$. This is a contradiction.

Remark 2. Although the algebra of the above example does not possess an orthonormal basis of geodesic vectors, it does possess an inner product for which there is a (nonorthonormal) basis of geodesic vectors. That is, we claim that there exists an inner product such that the span of all the geodesic vectors is the whole algebra. Using the above notation, regardless of the choice of an inner product, there is exactly one geodesic vector Y_1 (up to scaling) not lying in $\mathfrak{g}' = V_2$, namely any nonzero vector from $(\mathfrak{g}')^\perp$. We therefore want to show

that the span of all the geodesic vectors from \mathfrak{g}' covers \mathfrak{g}' . We will now specify the inner product. First choose X_2, X_3 to be orthonormal and orthogonal to V_4 . Then from case (4), a vector $Y = a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5$ (with $a_2 \neq 0$) is geodesic if it is orthogonal to V_4 (so $a_4 = a_5 = 0$) and $\sqrt{3}a_2 = \pm 2a_3$. This gives two linearly independent geodesic vectors $Y_2, Y_3 \in \mathfrak{g}' \cap V_4^\perp$, neither of which lies in V_3 . It remains to show that there exist at least two geodesic vectors in V_3 whose projections to V_4 are linearly independent. Define the remaining components of the inner product by requiring that $\|X_4\| = \|X_5\| = 1$ and $\langle X_4, X_5 \rangle = \varepsilon \in (0, 1)$. Then by case (3), a vector $Y = a_3X_3 + a_4X_4 + a_5X_5$ (with $a_3 \neq 0$) is geodesic if and only if the following two conditions hold.

$$4a_3^2 + a_4^2 - 2a_5^2 - \varepsilon a_4 a_5 = 0, \quad (2)$$

$$a_3(a_4 + \varepsilon a_5) + a_4(\varepsilon a_4 + a_5) = 0. \quad (3)$$

Note that $a_4 + \varepsilon a_5 \neq 0$, as otherwise from (3) either $a_4 = 0$ or $\varepsilon a_4 + a_5 = 0$ and in both cases we obtain $a_4 = a_5 = 0$ and then (2) would give $a_3 = 0$, which is a contradiction. Solving (3) for a_3 and substituting to (2) we get

$$4a_4^2(\varepsilon a_4 + a_5)^2 + (a_4^2 - 2a_5^2 - \varepsilon a_4 a_5)(a_4 + \varepsilon a_5)^2 = 0. \quad (4)$$

Note that as $a_3 \neq 0$ and $a_4 + \varepsilon a_5 \neq 0$, (3) gives $a_4 \neq 0$. Dividing (4) by a_4^4 and taking $t = a_5 a_4^{-1}$ we obtain

$$-2\varepsilon^2 t^4 - (4\varepsilon + \varepsilon^3)t^3 + (2 - \varepsilon^2)t^2 + 9\varepsilon t + (1 + 4\varepsilon^2) = 0.$$

The polynomial on the left-hand side has at least one positive root t_+ and at least one negative root t_- . It follows that V_3 contains at least two geodesic vectors $Y_\pm = (a_3)_\pm X_3 + X_4 + t_\pm X_5$ (where $(a_3)_\pm$ are determined by (3)). Their projections to V_4 are linearly independent, hence $\text{Span}(Y_1, Y_2, Y_3, Y_+, Y_-) = \mathfrak{g}$.

Note that for an arbitrary inner product on this algebra, the geodesic vectors may not span the entire algebra. For example, choosing an inner product with all the X_i 's orthogonal we obtain that the span of all the geodesic vectors is the proper subspace X_4^\perp of \mathfrak{g} .

We conclude this paper with two questions that have arisen from this study:

Question 1. Is it true that every unimodular Lie algebra possesses an inner product for which the geodesic vectors span the algebra?

Question 2. Apart from nilpotent Lie algebras, are there natural families of unimodular Lie algebras that possess an inner product for which there is an orthonormal basis of geodesic vectors?

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References

- [1] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation.
- [2] Grant Cairns, Ana Hinić Galić, and Yuri Nikolayevsky, *Totally geodesic subalgebras of nilpotent Lie algebras I*, preprint.
- [3] ———, *Totally geodesic subalgebras of nilpotent Lie algebras II*, preprint.
- [4] G. Calvaruso, O. Kowalski, and R. A. Marinosci, *Homogeneous geodesics in solvable Lie groups*, Acta Math. Hungar. **101** (2003), no. 4, 313–322.
- [5] Serena Cicalò, Willem A. de Graaf, and Csaba Schneider, *Six-dimensional nilpotent Lie algebras*, Linear Algebra Appl. **436** (2012), no. 1, 163–189.
- [6] Nathan Jacobson, *Lie algebras*, Dover Publications Inc., New York, 1979, Republication of the 1962 original.
- [7] V. V. Kaizer, *Conjugate points of left-invariant metrics on Lie groups*, Soviet Math. (Iz. VUZ) **34** (1990), no. 11, 32–44.
- [8] Oldřich Kowalski, Stana Nikčević, and Zdeněk Vlášek, *Homogeneous geodesics in homogeneous Riemannian manifolds—examples*, Geometry and topology of submanifolds, X (Beijing/Berlin, 1999), World Sci. Publ., River Edge, NJ, 2000, pp. 104–112.
- [9] Oldřich Kowalski and János Szenthe, *On the existence of homogeneous geodesics in homogeneous Riemannian manifolds*, Geom. Dedicata **81** (2000), no. 1-3, 209–214, Erratum, Geom. Dedicata, **84** (2001), no. 1-3, 331–332.
- [10] John Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329.
- [11] Klas Modin, Matthew Perlmutter, Stephen Marsland, and Robert McLachlan, *On Euler-Arnold equations and totally geodesic subgroups*, J. Geom. Phys. **61** (2011), no. 8, 1446–1461.
- [12] G. M. Mubarakzjanov, *On solvable Lie algebras*, Izv. Vysš. Učehn. Zaved. Matematika **1963** (1963), no. no 1 (32), 114–123.
- [13] Anthony Nielsen, *Fundamental differential geometry and geodesic flows on Lie groups*, MSc Thesis, La Trobe University, 2001.
- [14] W. V. Parker, *Sets of complex numbers associated with a matrix*, Duke Math. J. **15** (1948), 711–715.
- [15] H. Strade, *Lie algebras of small dimension*, Lie algebras, vertex operator algebras and their applications, Contemp. Math., vol. 442, Amer. Math. Soc., Providence, RI, 2007, pp. 233–265.
- [16] Hiroyuki Tasaki and Masaaki Umehara, *An invariant on 3-dimensional Lie algebras*, Proc. Amer. Math. Soc. **115** (1992), no. 2, 293–294.