Geometric Measure Theory

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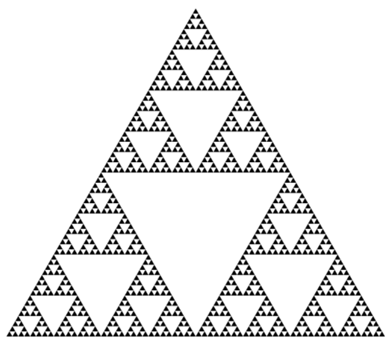
Though integration was discovered in the 17th century by Newton and Leibniz, it wasn't until around two centuries later that Riemann managed to give a rigorous definition of an integral. However, as time progressed and this definition was applied to stranger and more abstract situations, a number of shortcomings appeared and it was found to be inadequate.

Enter Lebesgue. To solve many of the problems that existed in Riemannian integration, Lebesgue laid out a two part theory. Firstly, he developed a theory of how to assign numbers to sets in a way that was compatible with our intuitive notions for length, area and volume. Then he used this foundation to build a theory of integration.



Geometric Measure Theory primarily concerns itself with the first part; expanding and understanding this method of giving sets length, area or volume.
The basic construction is that of the Hausdorff measure $\mathcal{H}^s$. The Hausdorff measure generalises volume in arbitrary dimension s. When one hears that fractals have "fractional dimension", it is usually referring to this notion. For example, the Sierpinski Triangle (right) has dimension approximately 1.585. In other words it means that this shape has finite $\mathcal{H}^{1.585}$ “volume", where as its 2-D area is 0 and the 1-D boundary is infinitely long.

Using this generalised notion of volume allows one to assign sensible values for volume to weird shapes, such as with the Sierpinski Triangle above, but it also opens up new theoretical tools for mathematicians to apply. The Area and Co-area formulae are core theorems in Geometric Measure Theory. The Area formula works by distorting one set into another whose volume can be more easily found, and the Co-area formula works by slicing a set into well understood pieces and then combining them together to find the total

These two processes might seem rather trivial, but the Co-area formula in particular is more than just a tool for calculation. It provides deep links between functional analysis and geometry. For example, crucial in establishing the existence of solutions of partial differential equations is the Sobolev Inequality. The Sobolev inequality bounds how fast a function can vary in terms of the size of the function itself. Approaching this inequality in a purely functional way is technical, and obtaining sharp bounds is very tricky.

However there is an inequality in geometry that has been known since antiquity and is intuitive to most people after a minutes thought. It is called the Isoperimetric Inequality. It says that for a fixed length of perimeter, out of all the shapes you could make, the circle encloses the most area. This holds in higher dimensions as well (obviously with the ball and its generalisations playing the role of the circle). The Co-area formula quickly shows these two inequalities are equivalent; the proof is well less than a page of working.

The other measure I studied in detail in my project is the perimeter measure. This is a highly technical construction. It works by using smooth functions to approximate potentially very jagged functions. But using it, one can show that all sets with ''finite perimeter" are well behaved. In particular, there is a powerful structure theorem for sets of finite perimeter that says that they have smooth boundaries, except perhaps at a small number of bad points. Such well-behaved-ness allows one to extend many familiar results of calculus to these often bizarre sets and the functions on them.

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