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Numerical Solutions of One-Dimensional Shallow Water Equations

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Introduction

This project contemplates numerical solutions to the non-linear Shallow Water Equations (SWE's) using a finite difference method for discretization of space and time variables. A linearization error is introduced for evaluating accurate numerical solutions. Accurate numerical solutions are obtained by efficient repositioning of mesh points for reducing the linearization error.

The SWE's are derived from accepted laws of physics conservation-of-momentum and hydro static laws. These are combined into a set of non-linear differential equations.

Solving the SWE's allows a person to derive two important unknown components, height of the wave and velocity, where height of the wave $h(x,t)$ is a function of space x and time t and similarly for the wave velocity $u(x,t)$.

Sets of time-stepped solutions are calculated numerically, subject to well defined initial conditions $u(x,0) = 0$, initial displacement $h(x,0) \ = 1 + \ \frac{2}{5}\exp(-5x^2)$, and boundary conditions for height $h(x_a,t) = 1$ and $h(x_b,t) = 1$, and for velocity $u(x_a,t) = 0$ and $u(x_b,t) = 0$.

Using the initial and boundary conditions, a set of time-stepped numerical solutions can be calculated to provide a set of solutions for the unknown velocity and height of a wave propagating through an incompressible media with a given constant density ρ.

This project applies the finite difference method to solve SWE's, in contrast to a closed form or exact solution using eigenvalues and eigenvectors [1] and an adaptive finite volume method [2].

This project demonstrates that finite difference methods can be used to solve SWE's and that further benefit can be obtained through mesh refinement and or optimisation, driven by error value targets. This benefit is a reduction of demand on memory size through efficient positioning of nodes based upon the error indicator.

Background to developing the one-dimensional SWE's

In deriving the one-dimensional shallow water equation, fluid (water) in a channel of unit width was contemplated. The vertical velocity of the water was assumed to be negligible and the horizontal velocity $u(x, t)$ as roughly constant throughout the channel cross section. This can be said to be true for small waves having a wave length greater than the depth. The fluid is assumed to be incompressible, so density ρ is constant. The depth of fluid given by $h(x, t)$ and velocity are variables for which we seek solutions.

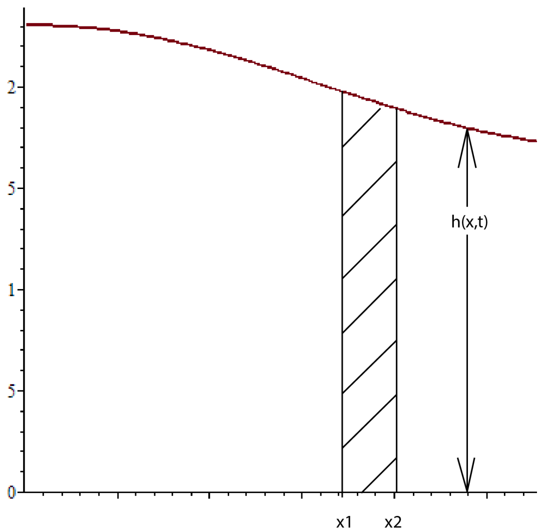

Figure 1: Graph of $h(x, t)$

The total mass in $x_1 \leq x \leq x_2$ at time t can be written as

$$\int_{x_1}^{x_2} \rho h(x, t)\, dx.$$

Density of momentum at each point is $\rho u(x,t)h(x,t)$, where the constant ρ drops out of the conservation-of-mass equation, which then takes the familiar form

$$h_t + [uh]_x = 0. \tag{1}$$

The conservation-of-momentum equation also takes the form of the gas dynamics equation

$$(\rho hu)_t + \left(\rho hu^2 + p\right)_x = 0.$$

Pressure p is determined by the hydrostatic law, stating the pressure at distance $h - y$ below the surface is $\rho g(h - y)$, where g is the gravitational constant. This pressure is as a result of the fluid weight above that particular point. Integrating over the interval $0 \leq y \leq h(x,t)$ gives the total pressure occurring at a particular point in space and time.

The correct pressure term in the momentum flux is

$$p = \frac{1}{2}\rho g h^2.$$

Using this form and cancelling ρ we get

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0. \tag{2}$$

Collecting the two equations (1) and (2) as a system of differential equations gives

$$\begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = 0. \tag{3}$$

Solving the Non-Linear Differential Equations Numerically

This project only considers a single hump of water as in [1], using an initial condition of $h(x,0) = 1 + \frac{2}{5}e^{-5x^2}$, and boundary conditions for height and velocity as in [1] of $-5 \leq x \leq 5$, $g = 1$ and $h(x_a,t) = h(x_b,t) = 1$, and $u(x_a,t) = u(x_b,t) = 0$.

The number of steps for space is N where $N \in \mathbb{Z}^+$ and the spatial index is i, $1 \leq i \leq N+1$. The step step size is $\delta = \frac{x_b - x_a}{N}$.

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Thus $x_i = x_a + (i-1)\delta$, $x_1 = x_a$ and $x_{N+1} = x_b$, and for N small intervals there will be $N-1$ nodes.

Similarly for time, the total number of steps are M where $M \in \mathbb{Z}^+$. For time t_j, $j \in \mathbb{Z}^+ : 1 < j \leq M$.

In developing the difference equations, forward difference for time and central difference for space has been used.

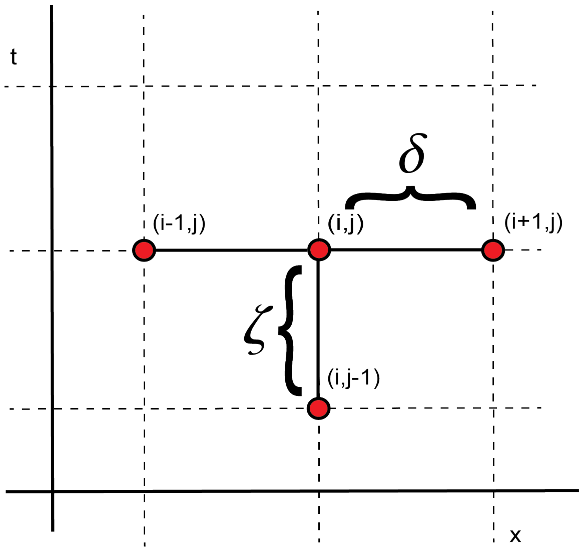

Figure 2: Mesh [3]

Developing the difference equations

Writing the corresponding difference equations of (1) and (2) term by term, as follows we assume that we know the values of $h(x,t)$ and $u(x,t)$ at time t_{j-1} and then find the values at time t_j.

Forward difference for time

$$\begin{aligned}\frac{d}{dt}h(x,t) &\Rightarrow \frac{h(x_i,t_j) - h(x_i,t_{j-1})}{\zeta} \\ &\Rightarrow \frac{h_{i,j} - h_{i,j-1}}{\zeta}\end{aligned}$$

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and

$$
\begin{aligned}
\frac{d}{dt}[h(x,t)u(x,t)] &= h(x,t)\frac{d}{dt}u(x,t) + u(x,t)\frac{d}{dt}h(x,t) \\
&\Rightarrow h(x_i,t_{j-1})\frac{d}{dt}u(x_i,t_{j-1}) + u(x_i,t_{j-1})\frac{d}{dt}h(x_i,t_{j-1}) \\
&\Rightarrow h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\zeta} + u_{i,j-1}\frac{h_{i,j}-h_{i,j-1}}{\zeta}.
\end{aligned}
$$

Central difference for space

$$
\begin{aligned}
\frac{d}{dx}[u(x,t)h(x,t)] &= u(x,t)\frac{d}{dx}h(x,t) + h(x,t)\frac{d}{dx}u(x,t) \\
&\Rightarrow u(x_i,t_{j-1})\frac{d}{dx}h(x_i,t_j) + h(x_i,t_{j-1})\frac{d}{dx}u(x_i,t_j) \\
&\Rightarrow u_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta}
\end{aligned}
$$

and

$$
\frac{d}{dx}\left[h(x,t)u^2(x,t) + \frac{1}{2}gh^2(x,t)\right] = \frac{d}{dx}\left[h(x,t)u^2(x,t)\right] + \frac{1}{2}g\frac{d}{dx}h^2(x,t)
$$

where

$$
\begin{aligned}
\frac{d}{dx}\left[h(x,t)u^2(x,t)\right] &= h(x,t)\frac{d}{dx}u^2(x,t) + u^2(x,t)\frac{d}{dx}h(x,t) \\
&= 2h(x,t)u(x,t)\frac{d}{dx}u(x,t) + u^2(x,t)\frac{d}{dx}h(x,t) \\
&\Rightarrow 2h_{i,j-1}u_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} + u^2_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} \\
&\Rightarrow h_{i,j-1}u_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{\delta} + u^2_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}
\end{aligned}
$$

and for the second term

$$
\begin{aligned}
\frac{d}{dx}\frac{1}{2}gh^2(x,t) &= 2\frac{1}{2}gh(x,t)\frac{d}{dx}h(x,t) \\
&= gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}.
\end{aligned}
$$

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The corresponding difference equations of (1) and (2) are as follows.

$$\begin{bmatrix} \frac{h_{i,j}-h_{i,j-1}}{\zeta} \\ h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\zeta} + u_{i,j-1}\frac{h_{i,j}-h_{i,j-1}}{\zeta} \end{bmatrix} + \begin{bmatrix} u_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} \\ h_{i,j-1}u_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{\delta} + u_{i,j-1}^2\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or for mass in discrete form

$$\frac{h_{i,j}-h_{i,j-1}}{\zeta} + u_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} = 0 \tag{4}$$

and momentum in discrete form

$$h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\zeta} + u_{i,j-1}\frac{h_{i,j}-h_{i,j-1}}{\zeta} + h_{i,j-1}u_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{\delta} + u_{i,j-1}^2\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} = 0 \tag{5}$$

Multiplying (4) by $u_{i,j-1}$

$$u_{i,j-1}\frac{h_{i,j}-h_{i,j-1}}{\zeta} + u_{i,j-1}^2\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + u_{i,j-1}h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} = 0 \tag{6}$$

Subtracting (4) from (5)

$$\begin{aligned} u_{i,j-1}\frac{h_{i,j}-h_{i,j-1}}{\zeta} + u_{i,j-1}^2\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + u_{i,j-1}h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} &= 0 \\ h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\zeta} + u_{i,j-1}h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} + gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} &= 0 \end{aligned} \tag{7}$$

Removing the $u_{i,j-1}^2$ component from (5) by dividing through with $u_{i,j-1}$

$$\begin{aligned} \frac{h_{i,j}-h_{i,j-1}}{\zeta} + u_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} &= 0 \\ h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\zeta} + u_{i,j-1}h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} + gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} &= 0 \end{aligned} \tag{8}$$

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Multiplying (4) and (5) by $2\delta\zeta$

$$\begin{aligned} 2\delta\left(h_{i,j}-h_{i,j-1}\right)+\zeta u_{i,j-1}\left(h_{i+1,j}-h_{i-1,j}\right)+\zeta h_{i,j-1}\left(u_{i+1,j}-u_{i-1,j}\right) &= 0 \\ 2\delta h_{i,j-1}\left(u_{i,j}-u_{i,j-1}\right)+\zeta u_{i,j-1}h_{i,j-1}\left(u_{i+1,j}-u_{i-1,j}\right)+\zeta g h_{i,j-1}\left(h_{i+1,j}-h_{i-1,j}\right) &= 0 \end{aligned}$$

Multiplying out the terms for (4)

$$\begin{aligned} 2\delta h_{i,j}-2 \quad \delta h_{i,j-1}+\zeta u_{i,j-1}h_{i+1,j}- \\ \zeta u_{i,j-1}h_{i-1,j}+\zeta h_{i,j-1}u_{i+1,j}-\zeta h_{i,j-1}u_{i-1,j}=0 \end{aligned} \tag{9}$$

and (5)

$$\begin{aligned} 2\delta h_{i,j-1}u_{i,j}-2\delta h_{i,j-1}u_{i,j-1}+\zeta u_{i,j-1}h_{i,j-1}u_{i+1,j}- \\ \zeta u_{i,j-1}h_{i,j-1}u_{i-1,j}+\zeta g h_{i,j-1}h_{i+1,j}-\zeta g h_{i,j-1}h_{i-1,j}=0 \end{aligned} \tag{10}$$

Moving the known terms to right side for (4)

$$\begin{aligned} 2\delta h_{i,j}+\zeta u_{i,j-1}h_{i+1,j}-\zeta u_{i,j-1}h_{i-1,j}+ \\ \zeta h_{i,j-1}u_{i+1,j}-\zeta h_{i,j-1}u_{i-1,j}=2\delta h_{i,j-1} \end{aligned} \tag{11}$$

and (5)

$$\begin{aligned} 2\delta h_{i,j-1}u_{i,j}+\zeta u_{i,j-1}h_{i,j-1}u_{i+1,j}- \\ \zeta u_{i,j-1}h_{i,j-1}u_{i-1,j}+\zeta g h_{i,j-1}h_{i+1,j}- \\ \zeta g h_{i,j-1}h_{i-1,j}=2\delta h_{i,j-1}u_{i,j-1} \end{aligned} \tag{12}$$

By way of example I take $N=4$, finding the initial solution for the discretized linear equations as follows:

For $i=1$

$$\begin{aligned} 2\delta h_{1,j}+\zeta u_{1,j-1}h_{2,j}- \\ \zeta u_{1,j-1}h_{0,j}+\zeta h_{1,j-1}u_{2,j}- \\ \zeta h_{1,j-1}u_{0,j}=2\delta h_{1,j-1} \end{aligned} \tag{13}$$

and

$$\begin{aligned} 2\delta h_{1,j-1}u_{1,j}+\zeta u_{1,j-1}h_{1,j-1}u_{2,j}- \\ \zeta u_{1,j-1}h_{1,j-1}u_{0,j}+\zeta g h_{1,j-1}h_{2,j}- \\ \zeta g h_{1,j-1}h_{0,j}=2\delta h_{1,j-1}u_{1,j-1} \end{aligned} \tag{14}$$

For $i = 2$

$$\begin{aligned} 2\delta h_{2,j} + \zeta u_{2,j-1} h_{3,j} - \\ \zeta u_{2,j-1} h_{1,j} + \zeta h_{2,j-1} u_{3,j} - \\ \zeta h_{2,j-1} u_{1,j} = 2\delta h_{2,j-1} \end{aligned} \tag{15}$$

and

$$\begin{aligned} 2\delta h_{2,j-1} u_{2,j} + \zeta u_{2,j-1} h_{2,j-1} u_{3,j} - \\ \zeta u_{2,j-1} h_{2,j-1} u_{1,j} + \zeta g h_{2,j-1} h_{3,j} - \\ \zeta g h_{2,j-1} h_{1,j} = 2\delta h_{2,j-1} u_{2,j-1} \end{aligned} \tag{16}$$

For $i = 3$

$$\begin{aligned} 2\delta h_{3,j} + \zeta u_{3,j-1} h_{4,j} - \\ \zeta u_{3,j-1} h_{2,j} + \zeta h_{3,j-1} u_{4,j} - \\ \zeta h_{3,j-1} u_{2,j} = 2\delta h_{3,j-1} \end{aligned} \tag{17}$$

and

$$\begin{aligned} 2\delta h_{3,j} + \zeta u_{3,j-1} h_{4,j} - \\ \zeta u_{3,j-1} h_{2,j} + \zeta h_{3,j-1} u_{4,j} - \\ \zeta h_{3,j-1} u_{2,j} = 2\delta h_{3,j-1} \end{aligned} \tag{18}$$

From the boundary conditions, the initial velocity is zero, namely $u(x_a, t) = u(x_b, t) = 0$, and the initial height is set to be $h(x_a, t) = h(x_b, t) = 1$, we find $u_{0,j} = 0$, $u_{4,j} = 0$, $h_{0,j} = 1$, and $h_{4,j} = 1$.

Therefore, the unknowns are $u_{1,j}$, $u_{2,j}$, $u_{3,j}$, $h_{1,j}$, $h_{2,j}$, $h_{3,j}$ and the above six equations can be simplified as:

For $i = 1$

$$\begin{aligned} 2\delta h_{1,j} + \zeta u_{1,j-1} h_{2,j} + \zeta h_{1,j-1} u_{2,j} &= 2\delta h_{1,j-1} + \zeta u_{1,j-1} \\ 2\delta h_{1,j-1} u_{1,j} + \zeta u_{1,j-1} h_{1,j-1} u_{2,j} + \zeta g h_{1,j-1} h_{2,j} &= 2\delta h_{1,j-1} u_{1,j-1} + \zeta g h_{1,j-1} \end{aligned}$$

For $i = 2$

$$\begin{aligned} 2\delta h_{2,j} + \zeta u_{2,j-1} h_{3,j} - \\ \zeta u_{2,j-1} h_{1,j} + \zeta h_{2,j-1} u_{3,j} - \\ \zeta h_{2,j-1} u_{1,j} = 2\delta h_{2,j-1} \end{aligned} \tag{19}$$

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and

$$
\begin{aligned}
2\delta h_{2,j-1}u_{2,j} + \zeta u_{2,j-1}h_{2,j-1}u_{3,j} - \\
\zeta u_{2,j-1}h_{2,j-1}u_{1,j} + \zeta g h_{2,j-1}h_{3,j} - \\
\zeta g h_{2,j-1}h_{1,j} = 2\delta h_{2,j-1}u_{2,j-1}
\end{aligned} \tag{20}
$$

For $i = 3$

$$
\begin{aligned}
2\delta h_{3,j} - \zeta u_{3,j-1}h_{2,j} - \zeta h_{3,j-1}u_{2,j} &= 2\delta h_{3,j-1} - \zeta u_{3,j-1} \\
2\delta h_{3,j-1}u_{3,j} - \zeta u_{3,j-1}h_{3,j-1}u_{2,j} - \zeta g h_{3,j-1}h_{2,j} &= 2\delta h_{3,j-1}u_{3,j-1} - \zeta g h_{3,j-1}
\end{aligned}
$$

Sorting and ordering the above six equations

$$
\begin{bmatrix}
0 & \zeta h_{1,j-1} & 0 & 2\delta & \zeta u_{1,j-1} & 0 \\
2\delta h_{1,j-1} & \zeta u_{1,j-1}h_{1,j-1} & 0 & 0 & \zeta g h_{1,j-1} & 0 \\
-\zeta h_{2,j-1} & 0 & \zeta h_{2,j-1} & -\zeta u_{2,j-1} & 2\delta & \zeta u_{2,j-1} \\
-\zeta u_{2,j-1}h_{2,j-1} & 2\delta h_{2,j-1} & \zeta u_{2,j-1}h_{2,j-1} & -\zeta g h_{2,j-1} & 0 & \zeta g h_{2,j-1} \\
0 & -\zeta h_{3,j-1} & 0 & 0 & -\zeta u_{3,j-1} & 2\delta \\
0 & -\zeta u_{3,j-1}h_{3,j-1} & 2\delta h_{3,j-1} & 0 & -\zeta g h_{3,j-1} & 0
\end{bmatrix}
\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ h_{1,j} \\ h_{2.j} \\ h_{3,j} \end{bmatrix} =
\begin{bmatrix}
2\delta h_{1,j-1} + \zeta u_{1,j-1} \\
2\delta h_{1,j-1}u_{1,j-1} + \zeta g h_{1,j-1} \\
2\delta h_{2,j-1} \\
2\delta h_{2,j-1}u_{2,j-1} \\
2\delta h_{3,j-1} - \zeta u_{3,j-1} \\
2\delta h_{3,j-1}u_{3,j-1} - \zeta g h_{3,j-1}
\end{bmatrix}
$$

Let

$$
A = \begin{pmatrix}
0 & \zeta h_{1,j-1} & 0 & 2\delta & \zeta u_{1,j-1} & 0 \\
2\delta h_{1,j-1} & \zeta u_{1,j-1}h_{1,j-1} & 0 & 0 & \zeta g h_{1,j-1} & 0 \\
-\zeta h_{2,j-1} & 0 & \zeta h_{2,j-1} & -\zeta u_{2,j-1} & 2\delta & \zeta u_{2,j-1} \\
-\zeta u_{2,j-1}h_{2,j-1} & 2\delta h_{2,j-1} & \zeta u_{2,j-1}h_{2,j-1} & -\zeta g h_{2,j-1} & 0 & \zeta g h_{2,j-1} \\
0 & -\zeta h_{3,j-1} & 0 & 0 & -\zeta u_{3,j-1} & 2\delta \\
0 & -\zeta u_{3,j-1}h_{3,j-1} & 2\delta h_{3,j-1} & 0 & -\zeta g h_{3,j-1} & 0
\end{pmatrix},
$$

$$
X = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ h_{1,j} \\ h_{2,j} \\ h_{3,j} \end{pmatrix},
$$

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$$b = \begin{pmatrix} 2\delta h_{1,j-1} + \zeta u_{1,j-1} \\ 2\delta h_{1,j-1} u_{1,j-1} + \zeta \\ 2\delta h_{2,j-1} \\ 2\delta h_{2,j-1} u_{2,j-1} \\ 2\delta h_{3,j-1} - \zeta u_{3,j-1} \\ 2\delta h_{3,j-1} u_{3,j-1} - \zeta g h_{3,j-1} \end{pmatrix},$$

The six equations now have the linear form $AX = b$ which can be solved, starting first with the known initial conditions.

Finite Difference Model Results

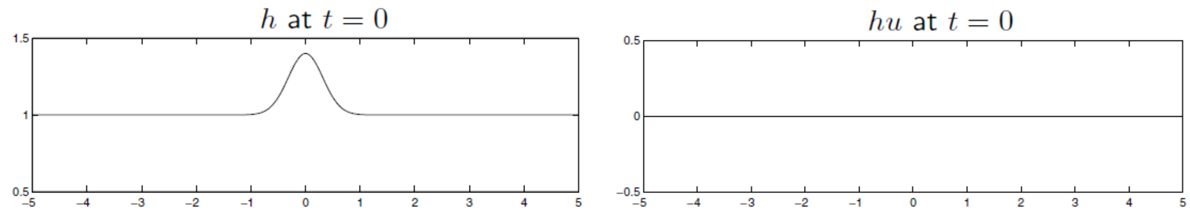

Figure 3: Graph of h and hu from [1] $t = 0$

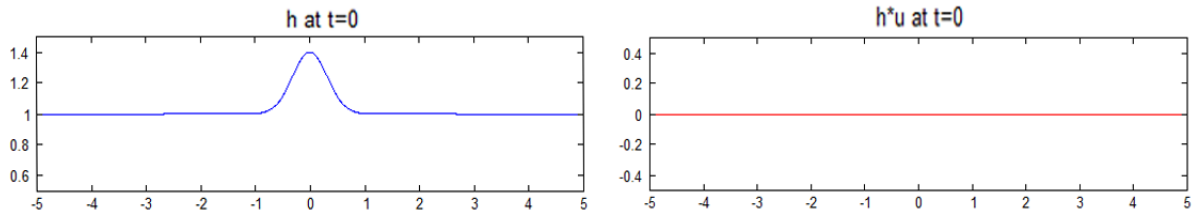

Figure 4: Graph of h and hu using finite difference $t = 0$

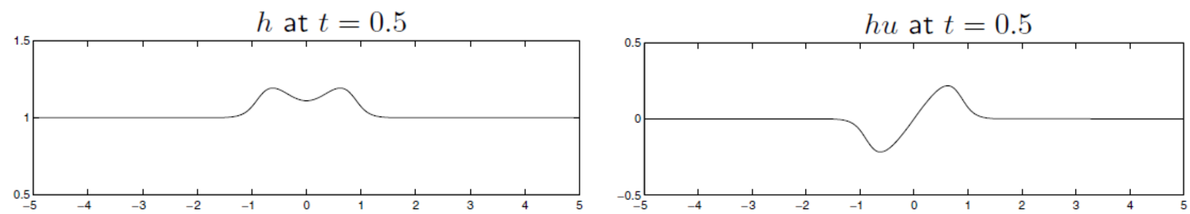

Figure 5: Graph of h and hu from [1] $t = 0.5$

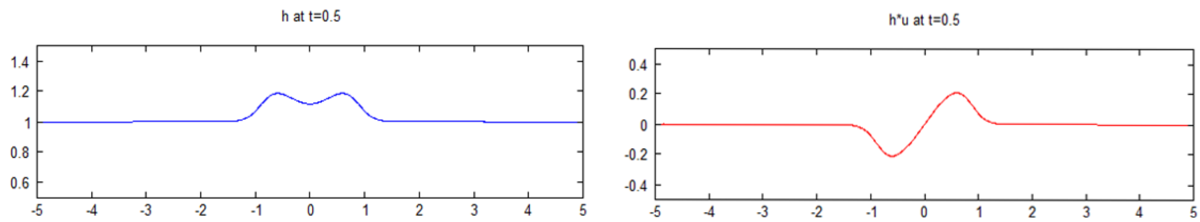

Figure 6: Graph of h and hu using finite difference $t = 0.5$

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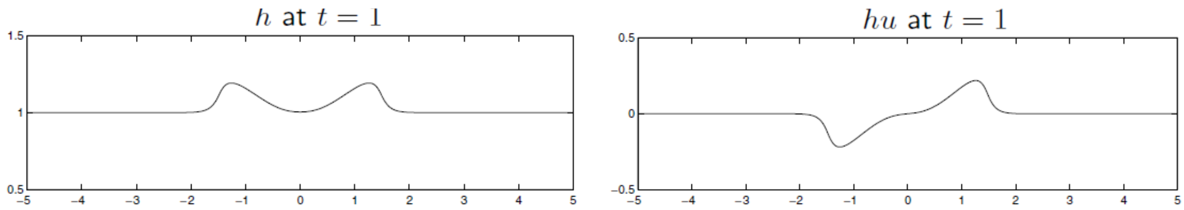

Figure 7: Graph of h and hu from [1] $t = 1$

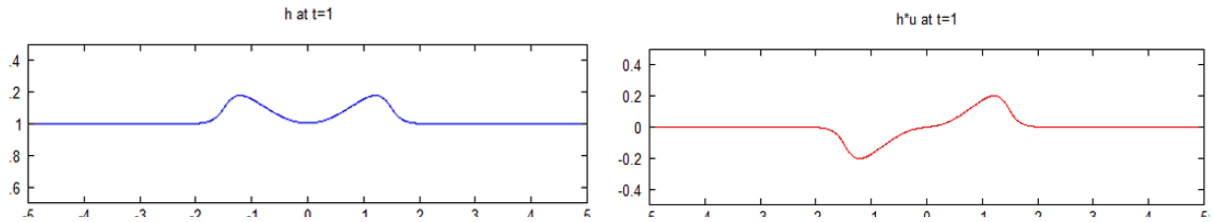

Figure 8: Graph of h and hu using finite difference $t = 1$

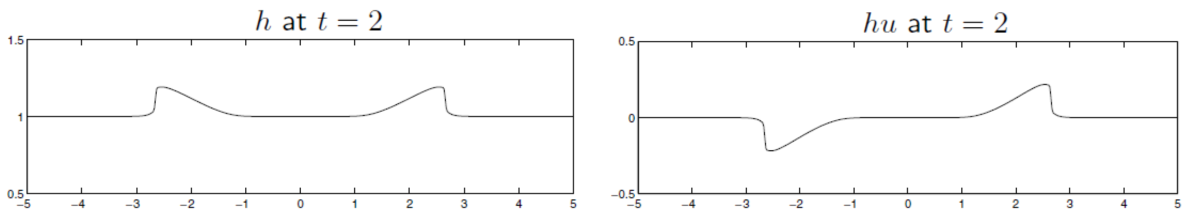

Figure 9: Graph of h and hu from [1] $t = 2$

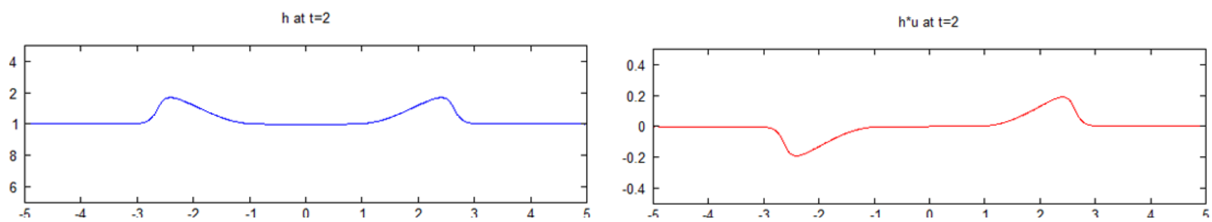

Figure 10: Graph of h and hu using finite difference $t = 2$

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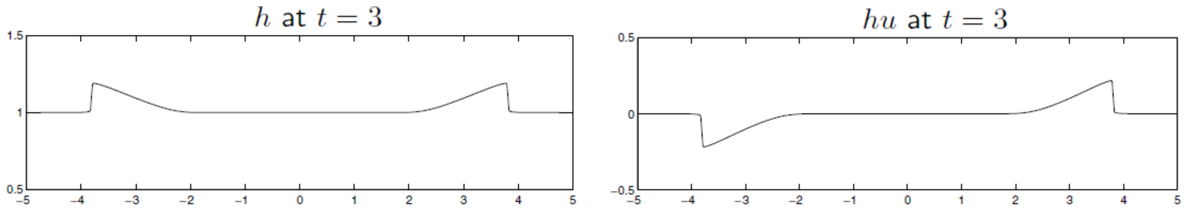

Figure 11: Graph of h and hu from [1] $t = 3$

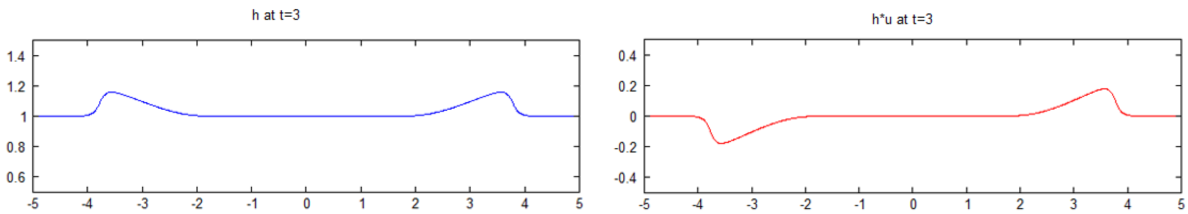

Figure 12: Graph of h and hu using finite difference $t = 3$

The accuracy of both h and hu profiles at $t = 0.5$ shown in Figure 6 are good by comparing with the corresponding results in Figure 5. The same accuracy is obtained for both h and hu profiles at $t = 1$ shown in Figure 8 by comparing with the corresponding results in Figure 7. However, the accuracy of both h and hu profiles at $t = 2$ and $t = 3$ are not good at the peak of waves by comparing with the corresponding profiles in Figure 9 and Figure 11. We introduce the approach of mesh refinement to improve the accuracy in the following section.

Approaches for Mesh Refinement

Mesh refinement, or node positioning, can occur globally for j, or locally around a given index j. The mesh refinement is based upon an error indicator at (i, j), namely the errors from the linearisation of the non-linear problem (3). This project progressed refinement for all i and j.

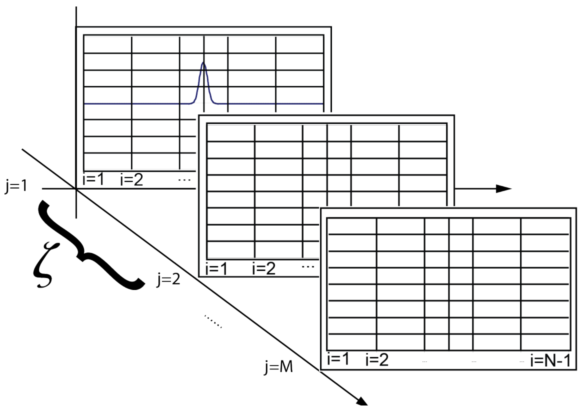

Figure 13: Stepped Solutions

As a finite difference, the error indicator is calculated in a different way, as follows.

Discrete (1) at (i, j) becomes

$$\frac{d}{dt}h(x,t) + u(x,t)\frac{d}{dx}h(x,t) + h(x,t)\frac{d}{dx} = 0$$

$$\frac{h_{i,j} - h_{i,j-1}}{\zeta} + u_{i,j}\frac{h_{i+1,j} - h_{i-1,j}}{2\delta} + h_{i,j}\frac{u_{i+1,j} - u_{i-1,j}}{2\delta} = 0 \quad (21)$$

and discrete (2) at (i, j) becomes

$$\frac{d}{dx}\left[u(x,t)h(x,t)\right] + \frac{d}{dx}\left[h(x,t)u(x,t)^2 + \frac{1}{2}gh(x,t)^2\right] = 0$$

$$\Rightarrow h_{i,j}\frac{u_{i,j} - ui,j-1}{\zeta} + u_{i,j}\frac{h_{i,j} - h_{i,j-1}}{\zeta} +$$

$$2h_{i,j}u_{i,j}\frac{u_{i+1,j} - ui-1,j}{2\delta} + u_{i,j}^2\frac{h_{i+1,j} - h_{i-1,j}}{2\delta} +$$

$$gh_{i,j}\frac{h_{i+1,j} - h_{i-1,j}}{2\delta} = 0$$

The error indicator checks the amplitude D of the differences between the left and right hand side of the equations (21) and (22) at (i, j). In this project, for simplicity

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all have been subdivided intervals into smaller even intervals if $D > \epsilon$, where ϵ is a pre-specified error tolerance. For example, we insert nine points evenly across $[-5, 5]$. If the error ϵ at one of the nine points is bigger than ϵ, then we subdivide the interval $[-5, 5]$ into twenty even intervals for a more accurate solution . This project only evaluated subdivision across the interval $[(i-1, j), (i, j)]$ and $[(i, j), (i+1, j)]$, such that if $D(i, j) > \epsilon$, then this interval would be subdivided into smaller.

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Mesh Refinement Results

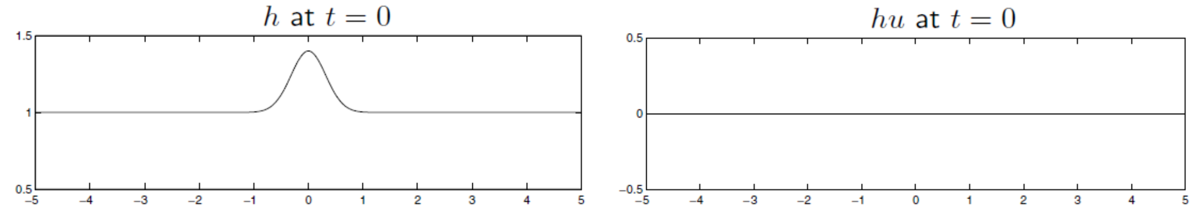

Figure 14: Graph of h and hu from [1] $t = 0$

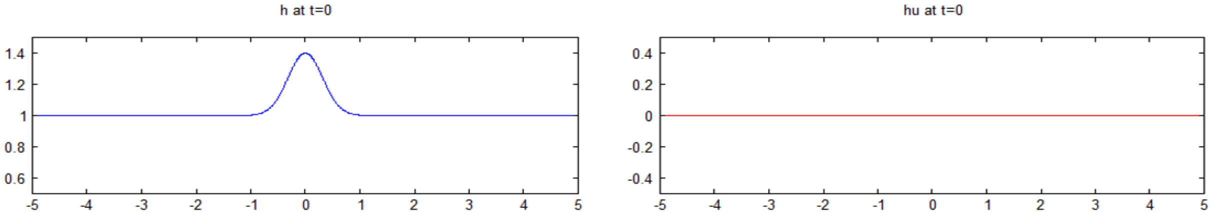

Figure 15: Graph of h and hu using finite difference $t = 0$

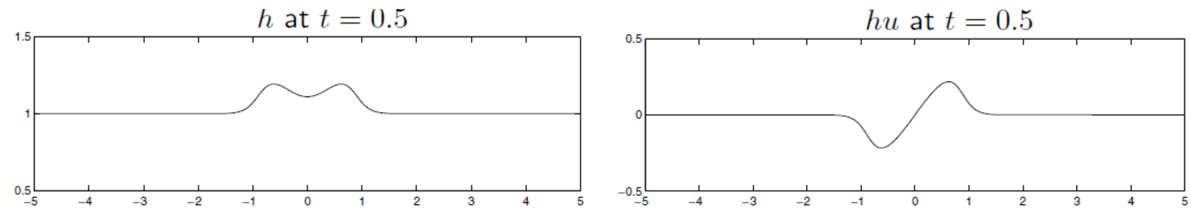

Figure 16: Graph of h and hu from [1] $t = 0.5$

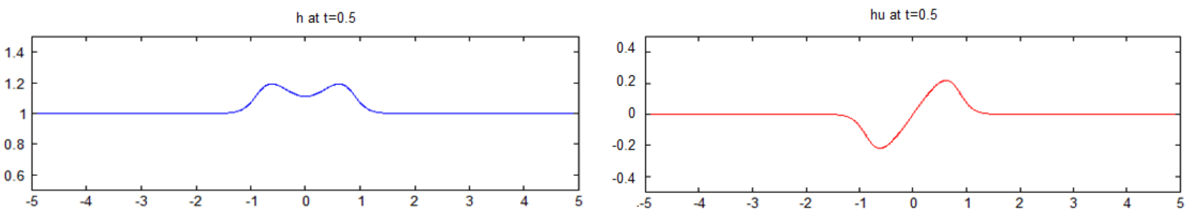

Figure 17: Graph of h and hu using finite difference $t = 0.5$

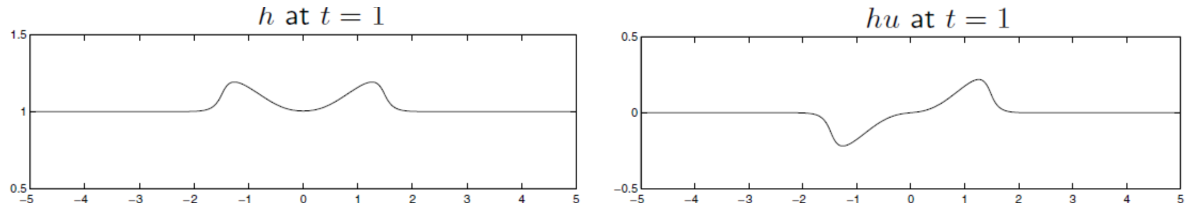

Figure 18: Graph of h and hu from [1] $t = 1$

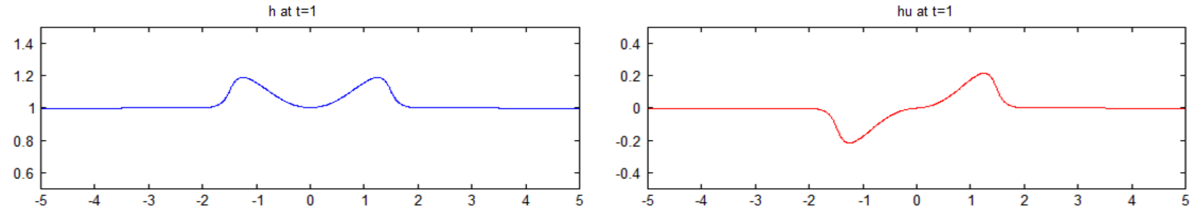

Figure 19: Graph of h and hu using finite difference $t = 1$

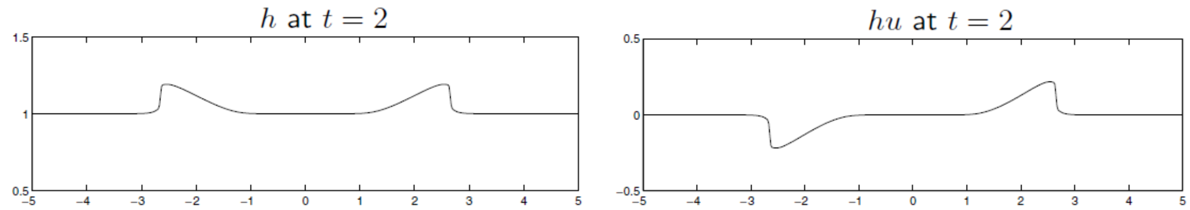

Figure 20: Graph of h and hu from [1] $t = 2$

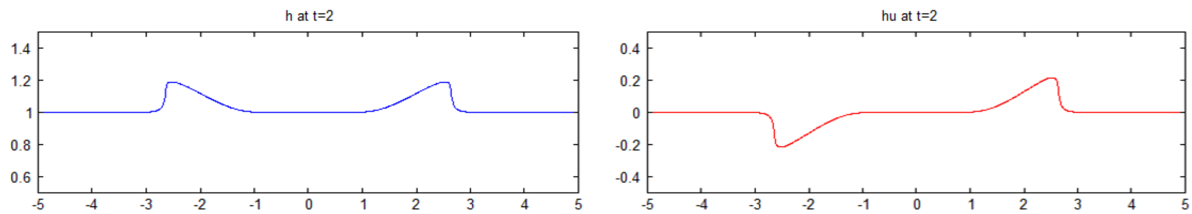

Figure 21: Graph of h and hu using finite difference $t = 2$

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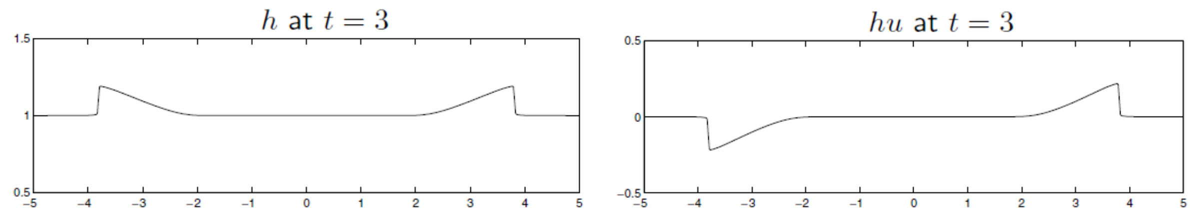

Figure 22: Graph of h and hu from [1] $t = 3$

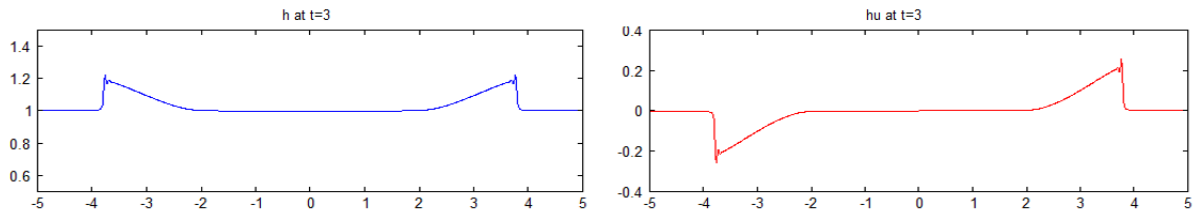

Figure 23: Graph of h and hu using finite difference $t = 3$

The accuracy of h and hu profiles at $t = 2$ shown in Figure 21 and $t = 3$ shown in Figure 23 have been improved comparing with those in Figure 20 and Figure 22. The profiles at $t = 3$ in Figure 23 are not smooth enough at the lowest and highest points of waves. This is one of our future research topics.

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Conclusion

Strong correlation was observed between the exact solution as used in [1] and the finite difference method used in this project. The finite difference method demonstrated itself as having capacity to providing good correlation between the closed form solution in [1] with the added advantage of providing optional mesh adjustment for improved accuracy.

Further Work

This work only considers SWE's in one dimension (1D), applying uniform refinement of the mesh. Further work would encompass local refinement of the mesh and expanding the finite difference method to 1.5D, 2D and eventually 3D, and applying mesh refinement to each.

AMSI Experience

A student's initial university experience provides exposure to academic staff through lectures and tutorials, but mostly lectures. Students are largely provided information in bulk to process and learn on an individual basis, at best collaboratively with other students similarly coming to terms with the same large amounts of information.

My experience and the opportunity to work first hand with senior academic staff has been refreshing and supportive. It provides insight on the post-graduate environment, which allows students to continue studies in areas of interest within a supportive environment.

I thank AMSI and CSU for the opportunity to work on this project, with special thanks to my supervisor.

References

[1] R. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge Texts in Applied Mathematics, (2004), Cambridge University Press.

[2] Sudi Mungkasi and Stephen Roberts, *Adaptive finite volume methods for the shallow water equations*, Department of Mathematics, Australian National University, A presentation in the 16th CTAC Conference Queensland University of Technology, 23-26 September 2012.

[3] John H. Mathews, *Numerical Methods For Mathematics, Science and Engineering*, Prentice Hall International Editions, 1992.

[4] Mahir Rasulov, Zafer Aslan and Ozkan Pakdill, *Finite differences method for shall water equations in a class of discontinuous functions* Department of Mathematics and Computing, Beykent University, Buykeekmece, Istanbul 34900, Turkey ELSEVIER Applied Mathematics and Computation 160 (2005) 343-353

[5] Sudi Mungkasi and Stephen G. Roberts, *Behaviour of the numerical entropy production of the one-and-a-half-dimensional shallow water equations*, 2 February, 2013

Appendix

The following MATLAB program is the implementation of the generalization of the above calculations for N=4 to any integer N.

Matlab Code for verification of the finite difference solution for $u(x,t)$ and $h(x,t)$

% Finite difference method for shallow water equations
% N+1: total number of points in interval [-5, 5] including the two ends, left and right
% left end $x_1 = -5$, and right end $x_(N+1) = 5$
% unknowns: $u[2:N]$ the velocity, $h[2:N]$ the depth of water
%
% Boundary conditions: $u(1,t) = u(N+1,t) = 0$; $h(1,t) = h(N+1,t) = 1$ where t=time
%
% Initial conditions:$h(x,0) = 1 + 2/5 * exp(-5 * x^2), u(x,0) = 0$
%
% delta: equal space $step = (5-(-5))/N$
% zeta: time step size
% Dimension of coefficient matrix A: $2N \times 2N$, b: $2N \times 1$ in $Ay = b$, $x = [uh]$
%
$clear$;
$N = 1000$; % number of steps in space
$zeta = 0.01$; % time step size
$M = 300$; % number of steps in time
% draw profile at $t = 0$; $t = 0.5$; $t = 1$; $t = 2$ and $t = 3$, the same as [1]

$g = 1$; % gravitational constant, the same as [1]
$delta = 10/N$;
$u(:,:) = zeros(N+1, M)$; % create space for velocity
$h(:,:) = zeros(N+1, M)$; % creat space for height
$x = -5 : delta : 5$; % space step and range
$u(1,:) = 0; u(N+1,:) = 0$; % velocity boundary conditions
$u(:,1) = 0$; %
$h(1,:) = 1$; $h(N+1,:) = 1$; % height boundary conditions
% initial displacement conditions
for $k = 2 : N$
$h(k,1) = 1 + 2/5 * exp(-5 * x(k)^2)$; %Initial conditions
end
% matrices for solving $A = zeros(2*(N-1), 2*(N-1)); b = zeros(2*(N-1), 1)$;

for $j = 2 : M$
$A(1, N) = 2 * delta; A(1, N + 1) = zeta * u(2, j - 1);$
$A(1, 2) = zeta * h(2, j - 1);$

%the first equation for$i = 1$ on page 5
$b(1, 1) = 2 * delta * h(2, j - 1) + zeta * u(2, j - 1) * h(1, j) + zeta * h(2, j - 1) * u(1, j);$
$A(2, 1) = 2 * delta * h(2, j - 1);$
$A(2, 2) = zeta * u(2, j - 1) * h(2, j - 1);$
%the second equation for i=1 on page 5
$A(2, N + 1) = zeta * g * h(2, j - 1);$
$b(2, 1) = 2 * delta * h(2, j - 1) * u(2, j - 1) + zeta * u(2, j - 1) * h(2, j - 1) * u(1, j) + zeta * g * h(2, j - 1) * h(1, j);$
$A(2 * N - 3, 2 * N - 2) = 2 * delta;$
$A(2 * N - 3, N - 2 + N - 1) = -zeta * u(N, j - 1);$
$A(2 * N - 3, N - 1 - 1) = -zeta * h(N, j - 1);$
%the first equation for i=3 on page 5
$b(2*N-3, 1) = 2*delta*h(N, j-1) - zeta*u(N, j-1)*h(N+1, j) - zeta*h(N, j-1)*u(N+1, j);$ $A(2 * N - 2, N - 1) = 2 * delta * h(N, j - 1);$
$A(2 * N - 2, N - 1 - 1) = -zeta * u(N, j - 1) * h(N, j - 1);$
$A(2 * N - 2, N - 2 + N - 1) = -zeta * g * h(N, j - 1);$
%the second equation for i=3 on page 5
$b(2 * N - 2, 1) = 2 * delta * h(N, j - 1) * u(N, j - 1) - zeta * u(N, j - 1) * h(N, j - 1) * u(N + 1, j) - zeta * g * h(N, j - 1) * h(N + 1, j);$
for $i = 3 : N - 1$
$A(2 * i - 3, N - 2 + i) = 2 * delta;$
$A(2 * i - 3, N - 2 + i + 1) = zeta * u(i, j - 1);$
$A(2 * i - 3, N - 2 + i - 1) = -zeta * u(i, j - 1);$
$A(2 * i - 3, i + 1 - 1) = zeta * h(i, j - 1);$
$A(2 * i - 3, i - 1 - 1) = -zeta * h(i, j - 1);$
%the first equation for$i = 2$ on page 5
$b(2 * i - 3, 1) = 2 * delta * h(i, j - 1);$
$A(2 * i - 2, i - 1) = 2 * delta * h(i, j - 1);$
$A(2 * i - 2, i + 1 - 1) = zeta * u(i, j - 1) * h(i, j - 1);$
$A(2 * i - 2, i - 1 - 1) = -zeta * u(i, j - 1) * h(i, j - 1);$
$A(2 * i - 2, N - 2 + i + 1) = zeta * g * h(i, j - 1);$
$A(2 * i - 2, N - 2 + i - 1) = -zeta * g * h(i, j - 1);$
%the second equation for $i = 2$ on page 5
$b(2 * i - 2, 1) = 2 * delta * h(i, j - 1) * u(i, j - 1);$ end
% solving y=A\ b;
% applying the solution to the velocity and height spaces
$u(2 : N, j) = y(1 : N - 1); h(2 : N, j) = y(N : 2 * N - 2);$

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```
end

figure
subplot(1,2,1); plot(x,h(:,1),'b-');
title('h at t=0');
axis([-5 5 0.5 1.5]);
subplot(1,2,2); plot(x,h(:,1).*u(:,1),'r-');
title('h*u at t=0');
axis([-5 5 -0.5 0.5]);

figure
subplot(1,2,1); plot(x,h(:,50),'b-');
title('h at t=0.5');
axis([-5 5 0.5 1.5]);
subplot(1,2,2); plot(x,h(:,50).*u(:,50),'r-');
title('h*u at t=0.5');
axis([-5 5 -0.5 0.5]);

figure
subplot(1,2,1); plot(x,h(:,100),'b-');
title('h at t=1');
axis([-5 5 0.5 1.5]);
subplot(1,2,2); plot(x,h(:,100).*u(:,100),'r-');
title('h*u at t=1');
axis([-5 5 -0.5 0.5]);

figure
subplot(1,2,1); plot(x,h(:,200),'b-');
title('h at t=2');
axis([-5 5 0.5 1.5]);
subplot(1,2,2); plot(x,h(:,200).*u(:,200),'r-');
title('h*u at t=2');
axis([-5 5 -0.5 0.5]);

figure
subplot(1,2,1); plot(x,h(:,300),'b-');
title('h at t=3');
axis([-5 5 0.5 1.5]);
subplot(1,2,2); plot(x,h(:,300).*u(:,300),'r-');
title('h*u at t=3');
axis([-5 5 -0.5 0.5]);
```

MatLab Code for Mesh Refinement

$forj = 2 : M$
$fori = 2 : N$
$Error1(i-1, j-1) = (h(i, j) - h(i, j-1))/zeta + u(i, j) * (h(i+1, j) - h(i-1, j))/(2 * delta) + h(i, j) * (u(i+1, j) - u(i-1, j))/(2 * delta);$
$Error2(i-1, j-1) = h(i, j) * (u(i, j) - u(i, j-1))/zeta + u(i, j) * (h(i, j) - h(i, j-1))/zeta + 2 * h(i, j) * u(i, j) * (u(i+1, j) - u(i-1, j))/(2 * delta) + (u(i, j))^2 * (h(i+1, j) - h(i-1, j))/(2 * delta) + g * h(i, j) * (h(i+1, j)$
$-h(i-1, j))/(2 * delta);$
end
end
$epsilon = 10^(- 2);$ % tolerance
$error = max(max(max(abs(Error1))), max(max(abs(Error2))))$
$iferror > epsilon$
$str = sprintf('Pleaseselectadif ferentNorasmallertimesteplessthan$
$\%dandrunthisprogramagain.', zeta);$
$disp(str);$

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