The Representation Theory of the Temperley-Lieb Algebras

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1 Introduction

The Temperley-Lieb algebras arise in the theory of quantum groups as endomorphism algebras of tensor powers of a distinguished representation of the quantised universal enveloping algebra $U_q(\mathfrak{sl}_2)$. They are also present in mathematical physics in the context of models for magnetism in statistical mechanics (such as the Heisenberg XXZ model). However, the Temperley-Lieb algebras admit a cellular structure which allows the study of its representations from a different perspective. Information gathered this way gives information about quantum groups and vice versa.

2 Algebras and Representations

The notion of an algebra plays a central role in what follows. An algebra over a field F is an F-vector space A, that is equipped with a distributive multiplication. We require that this multiplication has an identity element (which we will denote 1_A) and that it should be compatible with the scalar action of F on the vector space. That is, for all $a, b \in A$ and all $\lambda, \mu \in F$,

$$(\lambda a) \times (\mu b) = (\lambda\mu)\,(ab)$$

Many important examples of an F-algebras are endomorphism algebras of F-vector spaces. Given an F-vector space V, the endomorphism algebra of V (denoted $End_F(V)$) is the set of linear transformations from V to itself. Together with usual addition of linear maps and multiplication defined by composition, we obtain the structure of an F-algebra.

In the case where V is finite dimensional, we may choose a basis of V and identify the linear transformations with matrices; $End_F(V) \cong Mat_{\dim(V)}(F)$. Note that if V is one dimensional, $End_F(V)$ simply becomes F with the usual multiplication.

Similarly to many other algebraic structures, algebras have a notion of a homomorphism. An F-algebra homomorphism from A to B is an F-linear map $\varphi : A \to B$ which respects multiplication and identity elements:

$$\varphi(1_A) = 1_B$$

$$\varphi(a)\varphi(b) = \varphi(ab) \quad \text{for } a, b \in A$$

Two algebras are isomorphic if there is a bijective homomorphism between them, and quotients of algebras are defined analogously to quotients of rings. As one might suspect, the usual homomorphism theorems apply. If $\varphi : A \to B$ is an algebra homomorphism, then $A/\ker(\varphi) \cong im(\varphi)$.

A representation of an F-algebra A on a F-vector space V is a homomorphism ψ_V from A to $End_F(V)$. This can be thought of as concretely describing the abstract relations of the algebra with matrices. An important example for any algebra is the so-called left regular representation: take $V = A$, where each $a \in A$ acts via left multiplication on V.

Representations of an algebra can be combined in multiple ways. It is possible to "add" two representations ψ_V and ψ_W (on vector space V and W respectively) by taking the direct sum, $V \oplus W$, with the A-action defined by $\psi_{V\oplus W} = \psi_V \oplus \psi_W$

Additionally, it is possible to "multiply" vector spaces with the tensor product, which can be used to construct "products" of representations. The tensor product of two F-vector spaces V and W is another F-vector space denoted $V \otimes W$. It is defined by a universality property involving is a canonical bilinear map $\mu : V \times W \mapsto V \otimes W$ defined by $\mu(v, w) = v \otimes w$. This map has the property that if $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ are bases of V and W respectively, $V \otimes W$ has basis $\{v_i \otimes w_j\}_{(i,j)\in I\times J}$. In particular, $\dim(V \otimes W) = \dim(V) \times \dim(W)$.

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The next step for constructing tensor products of representations is to "multiply" linear transformations: If M and N are endomorphisms of V and W respectively, $(M \otimes N)(v \otimes w) = Mv \otimes Nw$ (for $v \in V$ and $w \in W$) defines an endomorphism of $V \otimes W$.

If A is an F-algebra, the multiplication $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$ gives $A \otimes A$ the structure of an F-algebra.

Finally, given two representations ψ_V and ψ_W of A, then $\psi_V \otimes \psi_W$ defines a representation of $A \otimes A$. This does not automatically become a representation of A. For this, we need a homomorphism $\Delta : A \to A \otimes A$. The composition $\Delta \circ (\psi_V \otimes \psi_W)$ then defines a representation of A. Such a homomorphism is called a comultiplication.

3 Quantum Groups

The theory of the Temperley-Lieb algebras is intimately connected to the quantum group $U_q(\mathfrak{sl}_2(F))$. Perhaps misleadingly, quantum groups are not groups, but associative algebras.

Given a field F, we let $F(q)$ to be the field of rational functions over F (in the variable q). Then, $U_q(\mathfrak{sl}_2(F))$ is the unital associative algebra over $F(q)$ with generators E, F, K, K^{-1}, subject to the following relations:

$$\begin{aligned} KE &= q^2 EK \\ KF &= q^{-2} FK \\ KK^{-1} = K^{-1}K &= 1 \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}} \end{aligned}$$

We will denote this algebra U. It is a "deformation" of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2(F)$. Although the above relations are not well defined if $q = 1$, by taking $q \to 1$ (with appropriate "algebraic scaffolding"), we recover the stated universal enveloping algebra. With similar techniques, it is possible to consider q equal to a definite value (rather than an indeterminate).

The algebra U has many properties. Like a universal enveloping algebra, it is a Hopf algebra; U has a comultiplication, antipode and counit, allowing the construction of tensor products of representations, dual representations and trivial representations, respectively. It is known that finite dimensional representations of U are semisimple (a direct sum of irreducible representations) when q is not a root of unity.

The algebra U is equipped with a comultiplication Δ:

$$\begin{aligned} \Delta(K) &= K \otimes K \\ \Delta(K^{-1}) &= K^{-1} \otimes K^{-1} \\ \Delta(E) &= E \otimes 1 + K \otimes E \\ \Delta(F) &= F \otimes K^{-1} + 1 \otimes F \end{aligned}$$

The most significant feature of Δ is that it is not symmetric under the exchange of the tensor factors. We say that the comultiplication is not cocommutative (this is contrast to the case of unversal enveloping algebras of lie algebras, as well as group algebras). However, it does satisfy a property called coassociativity. Coassociativity guarantees associativity of tensor products; for representations of U, $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$.

Due to the noncocommutativity of U, the linear map $P : V \otimes W \to W \otimes V$ which interchanges tensor factors ($v \otimes w \mapsto w \otimes v$) is not an isomorphism of representations of U. In this situation a suitable isomorphism is provided by so-called R-matrices which give rise to representations of braid groups, and appear in statistical mechanics.

There is a distinguished two dimensional irreducible representation $L(1,+)$ of U. It is spanned by m_0 and m_1, and with respect to this basis, the generators of U have the following matrices:

$$\begin{aligned} \psi_{L(1,+)}(K) &= \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \\ \psi_{L(1,+)}(K^{-1}) &= \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix} \\ \psi_{L(1,+)}(E) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \psi_{L(1,+)}(F) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

At last, we arrive at the definition of the Temperley-Lieb algebra. The set of linear maps of a representation V of U that commute with the U-action is denoted

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$End_U(V) = \{T \in End_F(V) \mid uTv = Tuv,\ \forall u \in U,\ \forall v \in V\}$. We consider the representation $V = L(1,+) \otimes L(1,+) \otimes L(1,+) \otimes \cdots \otimes L(1,+)$, where there are r tensor factors. This is written $V = L(1,+)^{\otimes r}$ for short.

The Temperley-Lieb algebra, written TL_r, is $End_U(L(1,+)^{\otimes r})$, for $r \in \mathbb{Z}^+$.

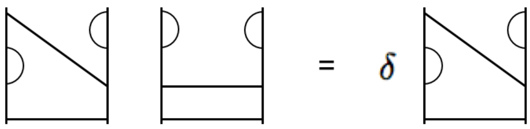
4 Cellular Algebra Structure

TL_r can also be described as a "diagram algebra". This involves giving a vector space basis of the algebra, together with diagrammatic multiplication rules for the basis elements.

Consider the set of diagrams consisting of two lines in a plane, each with distinguished r points, where pairs of points are connected with nonintersecting lines (up to homotopy equivalence). It can be shown easily that the number of such diagrams is the r^{th} Catalan number $\frac{1}{r+1}\binom{2r}{r}$ by constructing bijections with any of a selection of familiar combinatorial objects.

Diagrams can be multiplied by "concatenation"; the paths joining the second set of r points in the first diagram are connected to the first set of r points on the second diagram. For each closed loop in the new diagram, the outcome is multiplied by $\delta = q + q^{-1}$.

Here is an example calculation in TL_4:

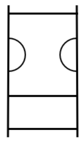


The algebra TL_r has a presentation in terms of generators $u_1, u_2, \cdots, u_{r-1}$, subject to the following relations:

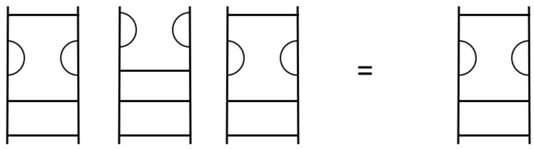
$$\begin{aligned} u_i u_{i\pm 1} u_i &= u_i \\ {u_i}^2 &= \delta u_i \\ u_i u_j &= u_j u_i \quad \text{for } |i-j| > 1 \end{aligned}$$

The i^{th} generator corresponds to the diagram where i^{th} and $(i+1)^{th}$ points are connected on the same sides ("semicircles"), and all other points are directly connected between the two sets of points. The pair of semicircles in each generator corresponds to a distinguished map on two tensor factors of $L(1,+)$ (of adjacent index). This map corresponds to the composition of a trace form on $L(1,+)$ with a map that takes an element x of F to x times the "quantum Casimir" for $L(1,+)^{\otimes 2}$ (analogous to the Casimir element in the theory of Lie algebras).

The generator u_2 in TL_5 is shown below.

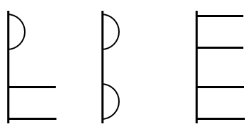


The relation $u_2u_1u_2 = u_2$ in TL_5 is shown below.



The Temperley-Lieb algebras can be studied using the theory of cellular algebras (which gives an elementary method of studying a large class of associative algebras).

Each diagram may be split down the middle to obtain "tableaux" with r points, some of which are connected by "arcs", others not being connected to any other point ("loose ends"). Additionally, every diagram arises by merging two tableaux with the same number of unconnected points (there is a unique way to join the opposing loose ends without them crossing over). Some examples of tableaux for TL_4 are below.



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Given any diagram equation in a Temperley-Lieb algebra, one can reflect the equation in a vertical axis (as if looking at it in a mirror) and still obtain a valid equation; the lines will reflect to connect the mirrored points, and the number of internal loops will not change. This corresponds to a linear anti-involution in the algebra (by mapping each diagram to its reflection and extending linearly). Together with aforementioned properties, this makes the Temperley-Lieb algebras into cellular algebras.

If we consider the vector space W with a basis consisting of tableaux, it is easy to see that the concatenation rule defines a representation of TL_r on this space (where a diagram concatenates with a tableaux to obtain another tableaux). However, the number of arcs on a tableaux basis element can never decrease under this action. This leads to the natural filtration $\{W_i\}$ of W, where W_i is spanned by tableaux with i or more arcs (each W_i contains the ones of larger index). Each W_i gives rise to an ideal of TL_r (corresponding to diagrams made from basis tableaux contained in W_i).

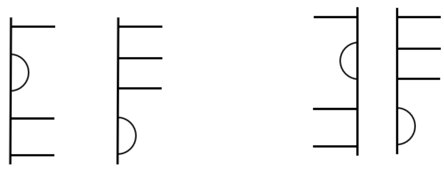
The structure of $L(1,+)^{\otimes r}$ is related to that of TL_r. By standard results from cellular theory, TL_r is semisimple (a sum of simple algebras) if and only if the representation on W_i/W_{i+1} is irreducible for each i. This in turn is equivalent to a particular matrix (with rows and columns indexed by tableaux corresponding to W_i/W_{i+1}) having nonzero determinant (the condition can also be stated in terms of the nondegeneracy of a particular bilinear form).

Given two tableaux, x, y, each with i arcs, the $(x,y)^{th}$ entry of the matrix is calculated by "multiplying" x and y as if they were Temperley-Lieb diagrams. If the loose ends of x do not connect to those of y, then the entry is zero. Otherwise, it is δ^k, where k is the number of closed loops created in performing the above computation.

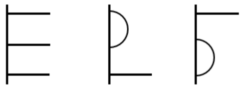
Here are example calculations for TL_5, in W_2 and W_1/W_2.

Here, the relevant matrix element will evaluate to δ, because the uppermost two points connect to form a loop, and the loose ends on either side connect to each other.

Here, the relevant matrix element will evaluate to zero, because the loose ends do not pair up from side to side; some are connected to other loose ends on the same side.



Such calculations can determine information about the structure of the Temperley-Lieb algebras. For example, consider TL_3. There are three tableaux, one with zero arcs, and two with one arc:



The W_0/W_1, the matrix is [1], due to the single tableaux with zero arcs. The determinant of this matrix is 1 (it is independent of δ), so this representation is always irreducible.

For W_1, the matrix has diagonal entires equal to δ and off-diagonal entries equal to 1. It is:

$$\begin{bmatrix} \delta & 1 \\ 1 & \delta \end{bmatrix}$$

The determinant is $\delta^2 - 1$.

This has zeroes $\delta = \pm 1$, so TL_3 is semisimple for all δ other than 1 and -1. Over $\mathbb{C}$, these values of $\delta = q + q^{-1}$ give $q = \exp(2\pi i/6), \exp(-2\pi i/6), \exp(2\pi i/3), \exp(-2\pi i/3)$; primitive third and sixth roots of unity. The same method may be applied in positive characteristic.

In this way, understanding the representations of $U_q(\mathfrak{sl}_2(F))$ can give combinatorial results about Temperley-Lieb diagrams, and vice versa.

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References

[1] Jantzen, J. C. 1996, *Lectures on Quantum Groups*, Graduate Studies in Mathematics 6, American Mathematical Society, Providence, RI.

[2] Graham, J. J. & Lehrer, G. I. (1996) Cellular Algebras, *Inventiones Mathematicae*, 123, 1-34.