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How many ways are there to tile a surface?

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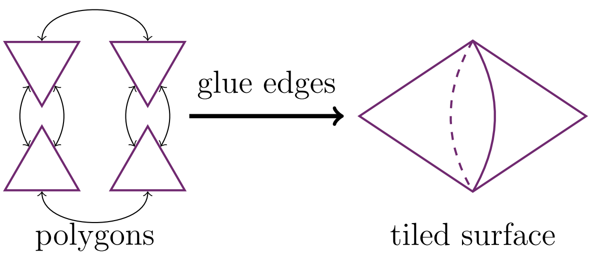
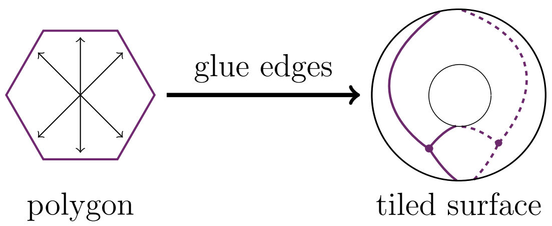
Introduction

Suppose we have a set of n of polygons. How many ways are there to make a surface by gluing the polygons' edges in pairs? We call this gluing a *tiling* of a surface, and we categorise these surfaces by the number of holes in them (i.e., their *genus*). It should be noted, by 'surface' we mean a 'closed, compact and orientable surface'.

For example, here is one possible tiling of a genus 0 surface using four triangles.



These tilings are not always so straightforward to visualize. For instance, we can tile a genus 1 surface using a single hexagon by gluing opposite edges in pairs.



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To make things easier, we a need a way to concretely enumerate all possible tilings and to be able to tell when two tilings are different and when they are the same.

Ribbon Graphs

The edges of the glued polygons of a tiling define a connected graph embedded on a genus g surface. This is equivalent to a graph equipped with a cyclic ordering of the edges around each of the vertices: that is, if we label the edges of the graph, the clockwise order of the edges around every vertex is sufficient to recover the tiled surface. We call this a *ribbon graph*. It should be noted that since we admit 1-sided and 2-sided polygons in the tiling, these graphs can have loops and multiple edges.

More formally, we can define ribbon graphs in terms of permutations. To do this, we first take a ribbon graph Γ and associate it with the set X consisting with the set of half-edges of the graph. We then have the following definition [2]

Definition 1. A **ribbon graph** is a triple (X, σ_0, σ_1) consisting of a finite set X, a permutation $\sigma_0 : X \to X$, and permutation $\sigma_1 : X \to X$ of cycle type $(2, 2, \ldots, 2)$. We require the group $\langle \sigma_0, \sigma_1 \rangle$ to act transitively on X, and the set $X/\langle \sigma_2 \rangle$ to be labelled from 1 up to n, where $\sigma_2 = \sigma_0\sigma_1$.

The primary benefit of defining ribbon graphs in terms of permutations is that it becomes possible to calculate all tilings computationally by exhaustively searching for triples $(\sigma_0, \sigma_1, \sigma_2)$ with the required properties.

We can also recover topological properties of a ribbon graph Γ from the permutation definition:

- $X_0 = X/\langle \sigma_0 \rangle$ is the set of vertices of Γ
- $X_1 = X/\langle \sigma_1 \rangle$ is the set of edges of Γ
- $X_2 = X/\langle \sigma_2 \rangle$ is the set of polygons used in the tiling (labelled from 1 to n in the definition)
- We can recover the genus using the formula for the Euler characteristic.

$$\begin{aligned} \#\text{ vertices} - \#\text{ edges} + \#\text{ faces} &= 2 - 2g \\ |X_0| - |X_1| + |X_2| &= 2 - 2g \end{aligned}$$

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Hypermaps

We now ask a more general question — how many ways can we make a genus g surface by gluing the edges of n polygons in around a-sided edges? A tiling using a-sided edges produces a graph embedded on a surface; we will call these *hypermaps*. In the $a = 2$ case, hypermaps are just ribbon graphs. Hypermaps generalize ribbon graphs and can be expressed analogously in terms of permutations.

Definition 2. Fix some integer $a \geq 2$. A **hypermap** is a triple (X, σ_0, σ_1) consisting of a finite set X, a permutation $\sigma_0 : X \to X$, and permutation $\sigma_1 : X \to X$ of cycle type $(a, a, \ldots, a)$. We require the group $\langle \sigma_0, \sigma 1 \rangle$ to act transitively on X, and the set $X/\langle \sigma_2 \rangle$ to be labelled from 1 up to n, where $\sigma_2 = \sigma_0 \sigma_1$.

It will be useful at this point to define automorphisms. An automorphism of a hypermap preserves the underlying structure of the tiling – the constituent polygons and the polygons each is attached to are unchanged. Automorphisms are, essentially, symmetries of the tiling. More formally, we have the following definition in terms of the permutations.

Definition 3. Let $\Gamma = (X, \sigma_0, \sigma_1)$ be a hypermap and $\sigma_2 = \sigma_0 \sigma_1$. An automorphism of Γ is a permutation ϕ of X such that:

1. $\phi \sigma_0 = \sigma_0 \phi$
2. $\phi \sigma_1 = \sigma_1 \phi$
3. $\phi \sigma_2 = \sigma_2$

The set of automorphisms of Γ form a group under composition, which is denoted by $\mathrm{Aut}(\Gamma)$.

We are now going to introduce some terminology. Let

$$M_{g,n}^{[a]}(b_1, b_2, \ldots, b_n)$$

denote the automorphism weighted count of genus g hypermaps made using a given set n polygons (a b_1-sided polygon, a b_2-sided polygon,..., and a b_n sided polygon) joined by a-sided edges. By 'automorphism weighted' we mean that a hypermap Γ will contribute $\frac{1}{|\mathrm{Aut}(\Gamma)|}$ to the enumeration. For instance, there is only one tiling of a genus 0 surface using a single rectangle whose edges are joined in pairs; however the hypermap corresponding to this tiling has two automorphisms, so

$$M_{0,1}^{[2]}(4) = \frac{1}{2}$$

The requirement that $\langle \sigma_0, \sigma 1 \rangle$ acts transitively on X ensures that the tiled surface is connected (i.e. ensures Γ is a single surface rather than two separate surfaces).

Enumeration

We can rephrase enumeration $M_{g,n}^{[a]}$ in terms of permutations using the above definitions. The number of half-edges of a hypermap representing a tiling is $|X| = \sum_{i=1}^{n} b_i$. So, to find all suitable hypermaps, we first need to find all triples $(\sigma_0, \sigma_1, \sigma_2)$ such that:

- the cycles of σ_2 correspond to the polygonal faces of the tiling. That is: σ_2 must have cycle type $(b_1, b_2, \ldots, b_n)$, such that $X_2 = X/\langle \sigma_2 \rangle$ recovers the set of polygons
- σ_1 has cycle type) $\underbrace{(a, a, \ldots, a)}_{\frac{|X|}{a} \text{ copies}}$, such that $X_1 = X/\langle \sigma_1 \rangle$ recovers the set of a-sided edges used in the tiling
- σ_0 has cycle type of length $2 - 2g - n + \frac{a-1}{a}|X|$, since $X_0 = X/\langle \sigma_0 \rangle$ is the set of vertices of the hypermap. This is a result from the Euler characteristic calculation of the tiled surface,

$$\begin{aligned} \#\text{ vertices} - \#\text{ edges} + \#\text{ faces} &= 2 - 2g \\ \#\text{ vertices} - \text{a}\times\#a\text{-sided edges} + \#\text{ polygons} + \#a\text{-sided edges} &= 2 - 2g \\ \#\text{ vertices} - a \cdot \frac{|X|}{a} + n + \frac{|X|}{a} &= 2 - 2g \end{aligned}$$

- $\sigma_0 \sigma_1 = \sigma_2$
- $\langle \sigma_0, \sigma_1 \rangle$ acts transitively on X

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Since the labelling of the elements of X is arbitrary, we divide the number of suitable triples $(\sigma_0, \sigma_1, \sigma_2)$ by all permutations of X, i.e. $|X|!$. This gives us the automorphism weighted count of hypermaps. That is:

$$M_{g,n}^{[a]}(b_1, b_2, \ldots, b_n) = \frac{1}{(b_1 + b_2 + \ldots + b_n)!} \cdot \#\text{tuples}(\sigma_0, \sigma_1, \sigma_2)$$

The following result from representation theory allows us to express the hypermap enumeration in terms of characters of the symmetric group. This is a very efficient way of calculating $M_{g,n}^{[a]}$ when $|X|$ is large.

Theorem 4 (Burnside Character Formula)**.** Let G be a group. Let the number of k-tuples $(\phi_1, \phi_2, \ldots, \phi_k)$, where each ϕ_i is in a given conjugacy class C_i of G, be denoted by $N(G : C_1, C_2, \ldots, C_k)$. Then:

$$N(G : C_1, C_2, \ldots, C_k) = \frac{\prod_{i=1}^{k} |C_i|}{|G|} \sum_{\rho} \frac{\prod_{i=1}^{k} \chi_\rho(\phi_i)}{\chi_\rho(1)^{k-2}}$$

where 1 is the group identity of G, the summation is over all irreducible representations ρ of G, and $\chi_\rho(\phi)$ denotes the evaluation of the character of the conjugacy class containing ϕ with respect to the irreducible representation ρ.

For the purpose of counting hypermaps, G is the symmetric group on b elements, where $b = \sum_{i=1}^{n} b_i$, and we want to count triples $(\sigma_0, \sigma_1, \sigma_2)$. The conjugacy classes of S_b consist of permutations with common cycle structure. This is convenient since, for a given enumeration $M_{g,n}^{[a]}(b_1, b_2, \ldots, b_n)$, we know the precise cycle structure of σ_1 and σ_2. However, we only know the number of cycles in σ_0, which complicates things. So, we require that σ_1 is in the conjugacy class corresponding to permutations with cycle structure $(a, a, \ldots, a)$, σ_2 is in the conjugacy class corresponding to permutations with cycle structure $(b_1, b_2, \ldots, b_n)$, and σ_0 is in one of many possible conjugacy classes corresponding to permutations with cycle type length $2 - 2g - n + \frac{b(a-1)}{a}$.

The Burnside character formula does not ensure that $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ forms a transitive subgroup of S_b, and therefore does not guarantee that the permutation triples it counts represent *connected* tiled surfaces. Consequently, we can only use this formula, in its present form, to compute $M_{g,n}^{[a]}$ when the cycle types of $\sigma_0, \sigma_1, \sigma_2$ alone ensure transitivity. There are two such cases; when any of $\sigma_0, \sigma_1, \sigma_2$ has exactly one cycle of length $|X|$ (this occurs when the tiling has a single vertex, or the tiling has a single

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a-sided edge and $|X| = a$, or the tiling consists of a single polygon with $|X|$ sides); or when no partition of any number in the cycle type of σ_1 appears in the cycle type of σ_0, and vice versa. For example, we could use the Burnside character formula if σ_1 has cycle type $(2, 2, 2)$ and σ_2 has cycle type $(3, 3)$; or when σ_1 has cycle type $(2, 2, 2)$ and σ_2 has cycle type (6).

More broadly, the Burnside character formula can be used to express the disconnected hypermap enumeration — the number of ways to tile a *disconnected* surface. This can, in fact, be used to to determine all connected hypermaps, and is a possible avenue for further work.

Recursion

One useful way of simplifying enumerations is to express difficult tilings in terms of simpler ones.For instance, we can construct ribbon graphs from simpler ribbon graphs in four distinct ways:

- joining two sides of a polygon by the addition of an edge to make two, smaller, polygons
- adding an edge to a polygon that is attached only at one edge
- adding an edge between two polygons belonging to a single ribbon graph
- adding an edge between two polygons belonging to different ribbon graphs

The above four cases were obtained by considering every type of ribbon graph produced by the removal of a single edge from a ribbon graph counted in $M^{[2]}_{g,n}(b_1, b_2, \ldots, b_n)$. By considering every possible construction of the above four forms, we overcount by exactly the number of edges in the ribbon graph. This leads to the following theorem:

Theorem 5. [3] By considering all the ways we can construct a ribbon graph from

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simpler ribbon graphs, we obtain a recursion for $M^{[2]}_{g,n}$:

$$\frac{b_1+b_2+\cdots+b_n}{2} M^2_{g,n}(b_1,b_2,\ldots,b_n) = \sum_{i<j}(b_i+b_j-2)M^2_{g,n-1}(b_{S\setminus\{i,j\}}, b_i+b_j-2)$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{p+q=b_i-2} pq\left[M^2_{g-1,n+1}(b_{S\setminus\{i\}},p,q)+\sum_{g_1+g_2=g}\sum_{I\sqcup J=S\setminus\{i\}} M^2_{g_1,|I|+1}(b_I,p)M_{g_2,|J|+1}(b_J,q)\right]$$

$$+\sum_{i=1}^{n}(b_i-2)M^2_{g,n}(b_{S\setminus\{i\}}, b_i-2)$$

For brevity, we have used $S=\{1,2,\ldots,n\}$ to simplify subscripts. For instance $(b_S)=(b_1,b_2,\ldots,b_n)$, and $(b_{S\setminus\{i\}})=(b_1,b_2,\ldots,b_{i-1},b_{i+1},\ldots,b_n)$

We can explain the coefficients in theorem 4 as follows; in the first case, there are b_i+b_j-2 ways to attach and edge to a face of perimeter b_i+b_j-2 to produce two faces of perimeters b_i and b_j. The summation is performed over all distinct pairs of faces b_i, b_j, so this construction contributes $\sum_{i<j}(b_i+b_j-2)M_{g,n-1}(b_{S\setminus\{i,j\}}, b_i+b_j-2)$ to the overcounted enumeration of $M_{g,n}(b_S)$. In the second case, there are $b-2$ ways to attach a edge at one end to a face of perimeter $b-2$, so construction contributes $\sum_{i=1}^{n}(b_i-2)M_{g,n}(b_{S\setminus\{i\}}, b_i-2)$. In the third and fourth cases, there are pq ways to attach an edge between faces of perimeters p and q. The summation for both of these must be over every choice of p and q such that $p+q=b_i-2$. The fourth case must also account for every choice of $g_1+g_2=g$ and all $I\sqcup J=S\setminus\{i\}$. Since the order of p,q is not relevant, we divide by 2, so that the third case contributes $\frac{1}{2}\sum_{i=1}^{n}\sum_{p+q=b_i-2} pqM_{g-1,n+1}(b_{S\setminus\{i\}},p,q)$. Similarly, since the order of g_1, g_2 is not relevant, fourth case contributes $\frac{1}{2}\sum_{i=1}^{n}\sum\limits_{p+q=b_i-2} pq \sum\limits_{g_1+g_2=g}\sum\limits_{I\sqcup J=S\setminus\{i\}} M_{g_1,|I|+1}(b_I,p)M_{g_2,|J|+1}(b_J,q)$.

Since every ribbon graph in the enumeration $M^2_{g,n}(b_S)$ could have been produced from any one of its edges in exactly one of the four ways listed above, the right hand side of the theorem overcounts the enumeration by a factor of $\frac{1}{2}\sum_{i=1}^{n} b_i$.

In a similar way, we can write a recursion for $M^{[3]}_{g,n}$ by considering all hypermaps produced by removing a single 3-sided edge.

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Theorem 6. We have the following recursion for $M_{g,n}^{[3]}$

$$\frac{1}{3}\sum_{i=1}^{n} b_i M_{g,n}^{[3]}(b_S) = \sum_{i=1}^{n}(b_i-3)M_{g,n}^{[3]}\left(b_{S\setminus\{i\}}, b_i-3\right) + \sum_{i=1}^{n}\binom{b_i-1}{3}M_{g-1,n}^{[3]}\left(b_{S\setminus\{i\}}, b_i-3\right)$$
$$+2\sum_{\substack{\text{distinct pairs}\\(i,j)}}(b_i+b_j-3)M_{g,n-1}^{[3]}\left(b_{S\setminus\{i,j\}}, b_i+b_j-3\right)$$
$$+2\sum_{\substack{\text{distinct triples}\\(i,j,k)}}(b_i+b_j+b_k-3)M_{g,n-2}^{[3]}\left(b_{S\setminus\{i,j,k\}}, b_i+b_j+b_k-3\right)$$
$$+\sum_{i=1}^{n}\sum_{p+q=b_i-3} pq\left[M_{g-1,n+1}^{[3]}\left(b_{S\setminus\{i\}}, p, q\right) + \sum_{g_1+g_2=g}\sum_{I\sqcup J=S\setminus\{i\}} M_{g_1,|I|+1}^{[3]}\left(b_I,p\right)M_{g_2,|J|+1}^{[3]}\left(b_J,q\right)\right]$$
$$+\sum_{\substack{\text{distinct pairs}\\(i,j)}}\sum_{p+q=b_i+b_j-3} pq\left[M_{g-1,n}^{[3]}\left(b_{S\setminus\{i,j\}}, p, q\right) + \sum_{g_1+g_2=g}\sum_{I\sqcup J=S\setminus\{i,j\}} M_{g_1,|I|+1}^{[3]}\left(b_I,p\right)M_{g_2,|J|+1}^{[3]}\left(b_J,q\right)\right]$$
$$+\sum_{i=1}^{n}\sum_{p+q+r=b_i-3} pqr\left[\frac{1}{3}M_{g-2,n+2}^{[3]}\left(b_{S\setminus\{i\}}, p, q, r\right) + \sum_{g_1+g_2=g-1}\sum_{I\sqcup J=S\setminus\{i\}} M_{g_1,|I|+2}^{[3]}\left(b_I,p,q\right)M_{g_2,|J|+1}^{[3]}\left(b_J,r\right)\right.$$
$$\left.+\frac{1}{3}\sum_{g_1+g_2+g_3=g}\sum_{I\sqcup J\sqcup K=S\setminus\{i\}} M_{g_1,|I|+1}^{[3]}\left(b_I,p\right)M_{g_2,|J|+1}^{[3]}\left(b_J,q\right)M_{g_3,|K|+1}^{[3]}\left(b_K,r\right)\right]$$

While this is faster than a brute force enumeration of $M_{g,n}^{[3]}$, this recursion does not generalize well. We can make a recursion for any choice of a, however, the size of this recursion becomes intractably large with increasing a; for $a = 2$ there are 4 terms; for $a = 3$ there are 11 terms; for $a = 4$ there are 41; and for $a = 5$ there are probably more than a hundred.

Eynard-Orantin Topological Recursion

The Eynard-Orantin recursion arose from studies in statistical mechanics and random matrix theory and has since found broader application in several other fields, such as combinatorial topology. We can make use of it to enumerate hypermaps.

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As an input, it takes a *spectral curve*

$$(x(z), y(z)) \in \mathbb{C}^2$$

where x and y are meromorphic functions and dx only has simple poles, and it produces a series of differential forms, $\omega_{g,n}(z_1, z_2, \ldots, z_n)$, called *Eynard-Orantin invariants.* The recursion is as follows:

Theorem 7 (Eynard-Orantin Recursion)**.** For a spectral curve $(x(z), y(z)) \in \mathbb{C}^2$, the base cases for the recursion are:

$$\omega_{0,1}(z_1) = ydx \qquad\qquad \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

and the recursion is:

$$\omega_{g,n}(z_1, z_S) = \sum_{\alpha} \operatorname*{Res}_{z=\alpha} \frac{-\int_{s_\alpha(z)}^{z} \omega_{0,2}(z_1, t)}{2(y(z) - y(s_\alpha(z)))dx} \Bigg[\omega_{g-1,n+1}(z, s_\alpha(z), z_{S\setminus\{1\}} + \sum_{\substack{g_1+g_2=g \\ I \bigsqcup J = S\setminus\{1\}}} \omega_{g_1,|I|+1}(z, z_I)\omega_{g_2,|J|+1}(s_\alpha(z), z_I)\Bigg]$$

where the sum is over all simple branch points α of dx, $s_\alpha(z)$ denotes a local involution around each branch point α, and I and J are non-empty.

Norbury [3], studied the spectral curve,

$$(x, y) = \left(z + \frac{1}{z}, z\right)$$

whose Eynard-Orantin variants, when expanded about $x_1 = x_2 = \ldots = x_n = \infty$, 'store' all the values of $M_{g,n}^{[2]}$. That is,

$$\omega_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{b_1, b_2, \ldots, b_n = 1}^{\infty} \frac{M_{g,n}^{[2]}(b_1, b_2, \ldots, b_n)}{x_1^{b_1+1} x_2^{b_2+1} \cdots x_n^{b_n+1}} dx_1 \, dx_2 \cdots dx_n$$

We can also use this spectral curve to express $M_{g,n}^{[2]}(b_1, b_2, \ldots, b_n)$ as a product of combinatorial factors and a quasi polynomial.

From the spectral curve it is possible to prove the following:

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Theorem 8. When $\sum_{i=1}^{n} b_i \equiv 0; (\mathrm{mod}\ 2)$, the following is true $M_{g,n}^{[2]}(b_1, b_2, \ldots, b_n) = C(b_1)C(b_2)\cdots C(b_n)Q_{g,n}(b_1, b_2, \ldots, b_n)$ where:

$$C(b) = \binom{b-1}{\lfloor (b-1)/2 \rfloor}$$

and $Q_{g,n}$ is a quasi polynomial modulo 2.

A quasi-polynomial modulo 2 is a polynomial whose terms depend on the congruence classes of its arguments modulo 2.

By calculating hypermap enumerations for other values of a, and searching for similar expressions, we arrive at the following conjecture.

Conjecture 9. When $\sum_{i=1}^{n} b_i \equiv 0; (\mathrm{mod}\ a)$, the following is true $M_{g,n}^{[a]}(b_1, b_2, \ldots, b_n) = C^{[a]}(b_1)C^{[a]}(b_2)\cdots C^{[a]}(b_n)Q_{g,n}^{[a]}(b_1, b_2, \ldots, b_n)$ where:

$$C(b)^{[a]} = \binom{b-1}{\lfloor (b-1)/a \rfloor}$$

and $Q_{g,n}^{[a]}$ is a quasi polynomial modulo a.

This also leads to another conjecture, that not only does a spectral curve exist for every hypermap enumeration, but that the spectral curve has a specific form.

Conjecture 10. The Eynard-Orantin invariants of the spectral curve

$$(x, y) = \left(\frac{z}{1+z^a}, z\right)$$

satisfy

$$\omega_{g,n}(x_1, x_2, \ldots, x_n) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n}\sum_{b_1,b_2,\ldots,b_n=1}^{\infty} M_{g,n}^{[a]}(b_1, b_2, \ldots, b_n)x_1^{b_1}x_2^{b_2}\ldots x_n^{b_n}dx_1\, dx_2\cdots dx_n$$

when expanded about $x_1 = x_2 = \ldots x_n$=0.

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Appendix: Calculations

Some arbitrarily chosen examples of hypermap enumerations.

g	n	$(b_1, b_2, \ldots, b_n)$	$M_{g,n}^{[2]}(b_1, b_2, \ldots, b_n)$
0	1	(2)	$\frac{1}{2}$
0	1	(4)	$\frac{1}{2}$
0	1	(8)	$\frac{7}{4}$
0	3	$(3, 2, 1)$	2
0	3	$(2, 2, 2)$	1
1	1	(4)	$\frac{1}{4}$
1	1	(6)	$\frac{5}{3}$
2	1	(8)	$\frac{21}{8}$
2	1	(12)	539

g	n	$(b_1, b_2, \ldots, b_n)$	$M_{g,n}^{[4]}(b_1, b_2, \ldots, b_n)$
0	1	(4)	$\frac{1}{4}$
0	1	(8)	$\frac{1}{2}$
0	1	(12)	$\frac{11}{6}$
0	2	$(2, 2)$	1
0	4	$(1, 1, 1, 1)$	6
0	4	$(2, 2, 2, 2)$	27
2	1	(24)	102393907
2	2	$(7, 1)$	117
2	2	$(5, 3)$	51

g	n	$(b_1, b_2, \ldots, b_n)$	$M_{g,n}^{[3]}(b_1, b_2, \ldots, b_n)$
0	1	(3)	$\frac{1}{3}$
0	1	(9)	$\frac{4}{3}$
0	3	$(1, 1, 1)$	2
0	3	$(3, 3, 3)$	8
1	1	(3)	$\frac{1}{3}$
1	1	(9)	$\frac{112}{3}$
1	2	$(4, 2)$	$\frac{7}{2}$
1	3	$(4, 4, 4)$	810
2	1	(30)	18153198063

g	n	$(b_1, b_2, \ldots, b_n)$	$M_{g,n}^{[5]}(b_1, b_2, \ldots, b_n)$
0	1	(5)	$\frac{1}{5}$
0	1	(10)	$\frac{1}{2}$
0	1	(15)	$\frac{7}{3}$
1	1	(5)	3
1	1	(10)	57
1	1	(15)	$\frac{2639}{3}$
1	2	$(4, 1)$	5
2	1	(10)	$\frac{12453}{10}$
2	2	$(5, 5)$	1210

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