Probability with Martingales for Economics and Finance

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Through my project I developed a theoretical understanding of martingales, before exploring the arbitrage-free pricing of a particular class of derivative contracts: barrier options.

Williams' classic text, *Probability with Martingales*, is an indispensable tool for the developing probabilist. Over summer, it served to unify my understanding of mathematical analysis and probability theory, while introducing a modicum of measure theory into my mathematical toolkit. Further, it ensured a constant challenge in the in-text problems (many of which I have left to solve!).

I applied learned theoretical knowledge from Williams to the financial problem of derivatives pricing, in particular attempting to find an analytic expression for the price of a specific type of barrier option, the up-and-in put.

It is interesting that a derivative contract is redundant in the sense that there exists a portfolio of stock and bond holdings, which can be updated through time at no cost to the owner, which precisely mimics the payoff of the required derivative contract. The price of the derivative today is equal to the construction cost of such a (replicating) portfolio, which can be shown to be an expectation of a (particular) random process with respect to the measure that makes it a martingale (Baxter & Rennie, 1996; Joshi, 2008).

Through modeling stock price evolution as an exponential Brownian motion, borrowing techniques from stochastic calculus, and making use of the reflection principal (Carr et al, 1998) we arrive at the analytic price for an up-and-in put option:

$$\begin{aligned} V(0) =& Ke^{-rT}\Phi\Big(\frac{\log(KS_0/B^2)-(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big)\cdot\exp(2(r-\tfrac{1}{2}\sigma^2)\log(B/S_0)\sigma^{-2}) \\ &- S_0\Phi\Big(\frac{\log(KS_0/B^2)-(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big)\cdot\exp(2(r+\tfrac{1}{2}\sigma^2)\log(B/S_0)\sigma^{-2}) \end{aligned}$$

where the value at time 0, V(0), is expressed in terms of a strike price *K*, interest rate *r*, time to maturity *T*, initial stock price S_0, stock volatility σ^2, barrier *B* and standard normal cumulative distribution function Φ.

The solution was verified using simulation techniques, coded in Matlab, which estimated the price by crude Monte Carlo, and provided statistical confidence interval bounds. The estimates agree with the analytic solution.

I derived many benefits from the AMSI summer research experience. I have spent summer exploring high-powered theory cognate to my future research interests. Beginning my Honors year with the momentum of 3 months of solid mathematics will ensure I start the university semester vigorously and with a large amount of confidence. Further, I have already had a chance to develop a working relationship with my Honors supervisor. I also learnt some of my own limitations as a mathematician over summer, which I hope to turn into strengths this year.
It was fruitful to combine scientific computing and Matlab coding with analytical pen-and-paper mathematics within the one project, a skill seeming increasingly important for the modern mathematician. Finally, it was beneficial to be given the chance to practice PowerPoint preparation and presentation skills for and at the CSIRO Big Day In 2011.

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