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Minimal Surfaces

Shuhui He

Supervised by Glen Wheeler & Valentina-Mira Wheeler

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Introduction

Minimal surfaces are beautiful geometric objects in surface theory. As the name suggests, they have least possible area subject to some constraint. They are realised in a variety of contexts, including soap films spanning a wire frame and the black hole horizon in general relativity. My project focuses on investigating elementary minimal surface theory, including several canonical examples of them and the Weierstrass representations. In particular, we first develop relevant background in differential geometry and partial differential equations (PDE), and use well-known examples of minimal surfaces such as Enneper's surface and catenoids to enhance our understanding. Then we turn to the main goal of the project, to establish the Weierstrass representations for minimal surfaces, which provides an effective way to generate examples of minimal surfaces and classify all of them. Future work involves investigating the application of a Weierstrass-like representation for biharmonic surfaces with a view towards new results on Chen's conjecture.

Preliminaries

In order to achieve the goal of this project, we need some background concepts from both differential geometry and complex analysis.

Regular Surface

In this work we only consider minimal surfaces that are regular surfaces in $\mathbb{R}^3$. We start with the definition of such a surface. Intuitively, a regular surface is a smooth surface, which allows us to perform basic calculus operations. It must have no sharp points, edges or self-intersections so that for every point on the surface, there exists a tangent plane. A precise definition is given by Do Carmo [1] as follows:

Definition 1. *A subset $S \subset \mathbb{R}^3$ is a regular surface if for each $p \in S$, there exists a neighbourhood $V \subset \mathbb{R}^3$ and a map $f : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that*

1. *f is differentiable.*
2. *f is a homeomorphism (continuous with continuous inverse).*
3. *the regularity condition holds. i.e., for each $q \in U$, the differential $df_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one to one.*

The mapping f is called a parametrisation or embedding.

In the above definition, condition 1 is necessary for performing differential calculus on surface S. Condition 2 has the purpose of preventing self-intersection and sharp points in S. It implies that the topology of S agrees with the subspace topology induced by the inclusion $S \subset \mathbb{R}^3$. Condition 3 states that the Jacobian of f is rank 2, that is one can find a 2×2 matrix inside the Jacobian which of full rank. This guarantees the existence of a tangent plane at any point $q(u_0, v_0) \in S$, where two linearly independent vectors $\partial_u f(q)$ and $\partial_v f(q)$ form a base of it.

Mean Curvature

To understand the nature of a surface, we need tools to describe how the surface bends. Curvature is a measurement of how much the surface bends towards the normal vector at a point p on the surface. A curvature at p in a tangential direction X is $\frac{1}{\rho}$ where ρ is the radius of the osculating circle at p tangential to X if the surface bends toward the normal vector at p and $\frac{-1}{\rho}$ otherwise. At each point p, the principal curvatures are the maximal and minimal curvatures through p, denoted by κ_1 and κ_2. We can hence define the mean curvature as follows

Definition 2. *The mean curvature of a surface S at p is*

$$H = \kappa_1 + \kappa_2.$$

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
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However, this definition is not easy to apply as the principal curvatures are not easy to compute. We will introduce two tools to help us.

Definition 3 (the First Fundamental Form). *[1] [4] The first fundamental form I of a surface element is the restriction of $\langle\, , \rangle$ to all tangent planes $T_p(S)$, that is,*

$$I(X, Y) := \langle X, Y \rangle \,.$$

In $\mathbb{R}^3$, it is denoted by the 2×2 matrix g_{ij}, where

$$g_{ij} = \langle \partial_i f, \partial_j f \rangle = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Definition 4 (the Second Fundamental Form). *The second fundamental form is the 2×2 matrix A_{ij}, given by*

$$A_{ij} = \langle \partial_i \partial_j f, \nu \rangle = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

where ν is a unit normal,

$$\nu(u, v) = \frac{\partial_u f \times \partial_v f}{|\partial_u f \times \partial_v f|}.$$

Now we can express the mean curvature H in terms of the coefficients of first and the second fundamental forms.

Lemma 5. *The mean curvature H can be expressed as*

$$H = \kappa_1 + \kappa_2 = trace\left(g^{ik} A_{kj}\right) = \frac{Eg + Ge - 2Ff}{EG - F^2}. \tag{1}$$

Using this formula, we can easily compute the mean curvature of a parametrised surface with parametrisation f. Notice that $H(p) = \phi(p)$ where $\phi : \Sigma \to \mathbb{R}^3$ is a partial differential equation.

Area Functional and Minimal Surface

We can define the surface area functional in terms of the first fundamental form.

Definition 6 (Surface Area). *Suppose $f : \Sigma \to S \subset \mathbb{R}^3$, the surface area of S is given by*

$$\begin{aligned} Area(f) &= \int_\Sigma |\partial_u f \wedge \partial_v f| \, du \, dv \\ &= \int_\Sigma \sqrt{EG - F^2} \, du \, dv \\ &= \int_\Sigma \sqrt{det \, g} \, du \, dv \\ &= \int_\Sigma d\mu. \end{aligned}$$

In order to study the local behaviour of the surface area functional, we look at the first variation of the functional, which is an analogue to the first derivative of a function.

Definition 7. *Let $f : \Sigma \to \mathbb{R}^3$ be a smooth closed immersed surface. A normal perturbation of f with speed $\eta : \Sigma \to \mathbb{R}$ is defined as*

$$\hat{f}(p, \epsilon) = f(p) + \epsilon\eta(p)\nu(p)$$

where ν is a choice of unit normal and $\epsilon \in \mathbb{R}$.

The first variation of f is expressed as

$$\lim_{\epsilon \to 0} \frac{\hat{f}(p, \epsilon) - f(p)}{\epsilon} = \frac{d}{d\epsilon} \hat{f}(p, \epsilon) \Big|_{\epsilon=0} .$$

Lemma 8. *Let $f : \Sigma \to \mathbb{R}^3$ be a smooth closed immersed surface. The first variation of surface area can be written in terms of the mean curvature H as follows:*

$$\frac{d}{d\epsilon} Area(\hat{f}) \Big|_{\epsilon=0} = -\int_\Sigma H\eta d\mu \, .$$

From the above lemma, we can see if $H = 0$, then irrespective of the speed η, the first variation of the area functional is zero. This means $H = 0$ is a critical point for the area functional. In order to check if it is a local minimum, maximum or saddle point, we need the second variation.

By now, we can see one reasonable way to define minimal surfaces is as following:

Definition 9 (Minimal Surface)**.** *A minimal surface is a surface S with mean curvature $H = 0$ at all points $p \in S$.*

Referring to equation (1), a minimal surface must fulfil the following PDE

$$\frac{Eg + Ge - 2Ff}{EG - F^2} = 0. \tag{2}$$

The above property is a property of the surface itself, regardless the parametrisation. However, a specific type of parametrisation is particularly useful in this work, which is known as an isothermal parametrisation.

Definition 10. *A parametrisation X is isothermal or conformal if*

$$E = \langle \partial_u X, \partial_u X \rangle = \langle \partial_v X, \partial_v X \rangle = G \text{ and } F = \langle \partial_u X, \partial_v X \rangle = 0.$$

With such a parametrisation, we can simplify equation (2) and have the following result

Lemma 11. *Let S be a surface with isothermal parametrisation. Then S is minimal if and only if $e = -g$.*

One concern is that some common parametrisations of minimal surfaces are not isothermal, but the following theorem shows that requiring minimal surfaces to have an isothermal parametrisation is not an essential restriction.

Theorem 12. *Every minimal surface in $\mathbb{R}^3$ has a locally isothermal parametrisation.*

Complex Analysis

Several concepts from complex analysis are essential for the study of the Weierstrass Representation for minimal surfaces.

Definition 13 (Holomorphic Function)**.** *Suppose f is a function of a complex variable and that $a \in \mathbb{C}$. Suppose also that some neighbourhood of a lies within the domain of definition of f. Then the derivative of f at a is the limit*

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}.$$

We say that f is complex differentiable at a if the limit exists. If f is complex differentiable at every point a in an open set D, we say that f is holomorphic on D.

If a complex function $f(z) = f(u+iv) = x(u,v) + iy(u,v)$ is holomorphic, then x and y have first partial derivatives with respect to u and v, and satisfy the Cauchy-Riemann equations:

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} \qquad \text{and} \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u},$$

or, equivalently, the *Wirtinger derivative* of f with respect to the complex conjugate of z is zero:

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Definition 14 (Wirtinger Derivatives)**.** *Let f be a function of one complex variable having the form $f(z) = x(u,v) + iy(u,v)$, the Wirtinger derivatives are defined as*

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} - \frac{\partial x}{\partial v}\right),$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial x}{\partial u} - \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} + \frac{\partial x}{\partial v}\right).$$

We call $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ the Wirtinger operators.

Definition 15 (Harmonic Functions)**.** *Let f be a function of one complex variable having the form $f(z) = x(u,v) + iy(u,v)$, f is called harmornic if $\partial_{uu} f + \partial_{vv} f = 0$.*

Definition 16 (Meromorphic Function)**.** *Suppose $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, f is said to be meromorphic if there exists isolated points z_i, such that $\tilde{f} : \Omega \backslash \{z_i\} \to \mathbb{C}$ is holomorphic.*

In addition, the points $\{z_i\}$ are called poles of the function. The order of the pole z_i is the number of terms that vanish in the Laurent series expansion around z_i.

Weierstrass Representations for Minimal Surfaces

The study of minimal surfaces originates with Lagrange in 1776. He derived a PDE to describe such surfaces, but he did not succeed in finding any solution except for the plane. Progress in the study of minimal surfaces was slow. For almost a hundred years, only a few more examples such as the catenoid and Scherk's surfaces were found. Motivated by the classification and generation of examples, in the 1860s Weierstrass and Enneper developed some useful representation formulae linking minimal surfaces to complex analysis.

Minimal surfaces of revolution are particularly easy to classify as the theorem below demonstrates:

Theorem 17. *[3] Each complete minimal surface of revolution is either a plane or a catenoid*

$$f : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}^3, f(t, \theta) := (x(t) \cos\theta, x(t) \sin\theta, t)$$
$$\textit{with } x(t) := a \cosh\left(\frac{t - t_0}{a}\right) \textit{ for } t_0 \in \mathbb{R},\ a > 0. \tag{3}$$

Proof. We first recall that any surface of revolution is obtained by rotating a generating curve $(x, z) : I \to (0, \infty) \times \mathbb{R}$ in the xz-plane about the z-axis. It can be parametrised by

$$f : I \times \mathbb{R} \to \mathbb{R}^3,\ f(t, \theta) := (x(t) \cos\theta, x(t) \sin\theta, z(t)) .$$

Then, the zero mean curvature equation (2) can be rewritten as an ODE with respect to t,

$$x\left(x'z'' - z'x''\right) + z'\left(x'^2 + z'^2\right) = 0.$$

It is easy to check that for the special case $h \equiv \textit{const}$ the above equation is satisfied. This corresponds to a horizontal plane without the point $P := (0, 0, z)$ and it becomes complete by taking the union with P.

Now we consider the case when h is not identically constant. Then there is $t_0 \in I$ with $h'(t_0) \neq 0$. Hence, h is locally monotone, and by a re-parametrisation of the generating curve we may assume $h(t) = t$ locally. This helps us to simplify down the minimal surface equation to

$$xx'' = x'^2 + 1, \qquad x > 0. \tag{4}$$

We solve this ODE by using the substitution $x' = y$, then separation of variables:

$$\begin{aligned}(4) \implies xy' &= y^2 + 1\\ \frac{1}{x} &= \frac{y'}{y^2 + 1}\\ \ln(x) &= \ln\left(y^2 + 1\right)^{\frac{1}{2}} + C_0\\ x^2 &= C_1\left(x'^2 + 1\right).\end{aligned}$$

It is not hard to check that (3) is a solution to the ODE, with $C_1 = a^2$.

We should now check if this is the only solution. Notice that the ODE system $x' = y$ and $y' = (1 + y^2)/x$ is Lipschitz for $r > \epsilon > 0$ and so this system satisfies the Picard-Lindelöf theorem, hence the solution is unique. Moreover, this family of solution are defined for all $t \in \mathbb{R}$ and hence complete.

Now we can conclude that all solutions are of the type claimed.

□

Postal Address: 111 Barry Street
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Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

From now on, we will be focusing on establishing the Weierstrass Representations with our existing foundation. This involves combining complex analysis and classical minimal surface theory.

Theorem 18. *If the parametrisation X is isothermal, then*

$$\partial_{uu} X + \partial_{vv} X = EH\nu,$$

where E is coefficient of the first fundamental form, H is mean curvature and ν is an unit normal.

Proof. [2] The set $\{\partial_u X, \partial_v X, \nu\}$ forms an orthonormal basis for $\mathbb{R}^3$. We can express the vector $\partial_{uu} X$ and $\partial_{vv} X$ in terms of these bases vectors. That is,

$$\begin{aligned}\partial_{uu} X &= \Gamma^u_{uu} \partial_u X + \Gamma^v_{uu} \partial_v X + \langle \partial_{uu} X, \nu \rangle \nu, \\ \partial_{vv} X &= \Gamma^u_{vv} \partial_u X + \Gamma^v_{vv} \partial_v X + \langle \partial_{vv} X, \nu \rangle \nu.\end{aligned}$$

Where $\Gamma^u_{uu}, \Gamma^v_{uu}, \Gamma^u_{vv}$ and Γ^v_{vv} are known as Christoffel symbols, indicating the tangential projection of the second derivative. i.e., $\Gamma^k_{ij} \partial_k X = (\partial_i \partial_j X)^T$.

By comparing $\partial_u E$ to $\langle \partial_{uu} X, \partial_u X \rangle$, we have

$$\begin{aligned}\frac{1}{2} \partial_u E = \langle \partial_{uu} X, \partial_u X \rangle &= \Gamma^u_{uu} \langle \partial_u X, \partial_u X \rangle + \Gamma^v_{uu} \langle \partial_v X, \partial_u X \rangle + 0 \\ &= \Gamma^u_{uu} E + \Gamma^v_{uu} F \\ &= \Gamma^u_{uu} E,\end{aligned}$$

hence

$$\Gamma^u_{uu} = \frac{\partial_u E}{2E}.$$

Similarly, we can derive all four Christoffel symbols mentioned above, and obtains

$$\begin{aligned}\partial_{uu} X &= \frac{\partial_u E}{2E} \partial_u X - \frac{\partial_v E}{2G} \partial_v X + e\nu, \\ \partial_{vv} X &= -\frac{\partial_u G}{2E} \partial_u X + \frac{\partial_v G}{2G} \partial_v X + g\nu.\end{aligned}$$

Using the mean curvature equation (1) and properties of isothermal parametrisation, it is now straight forward to show that $\partial_{uu} X + \partial_{vv} X = EH\nu$.

□

Corollary 19. *A surface S with an isothermal parametrisation $X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ is minimal $\iff$ x_1, x_2, x_3 are harmonic functions.*

Proof. From Theorem 18, we observe that $\partial_{uu}X + \partial_{vv}X = 0$ if and only if $H = 0$, since both $E \neq 0$ and $\nu \neq 0$. Hence the coordinate functions are harmonic if and only if $H = 0$, implying S is minimal.

□

Lemma 20. *Let f be a function of one complex variable having the form $f(z) = x(u,v) + iy(u,v)$, we have*

$$4\left(\frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}\right)\right) = \partial_{uu}f + \partial_{vv}f. \tag{5}$$

Proof. This result can be easily proved by applying the Wirtinger derivatives on LHS and the usual derivatives on the RHS.

□

Theorem 21. *[2] Let S be a surface with parametrisation $X = (x_1, x_2, x_3)$ and let $\phi = (\varphi_1, \varphi_2, \varphi_3)$, where $\varphi_k = \frac{\partial x_k}{\partial z}$. Then X is isothermal $\iff$ $\phi^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$. If X is isothermal, then S is minimal $\iff$ each x_k is harmonic $\iff$ each φ_k is holomorphic.*

Proof. We compute $(\varphi_k)^2$ by applying the Wirtinger operators. As all x_k are real functions, the complex components are simply zero.

$$\begin{aligned}
(\varphi_k)^2 &= \left(\frac{\partial x_k}{\partial z}\right)^2 \\
&= \left[\frac{1}{2}\left(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v}\right)\right]^2 \\
&= \frac{1}{4}\left[\left(\frac{\partial x_k}{\partial u}\right)^2 - \left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right].
\end{aligned}$$

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c/- The University of Melbourne
Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
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Hence we obtain

$$\begin{aligned}\phi^2 &= (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 \\ &= \frac{1}{4}\left[\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2 - \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\sum_{k=1}^{3}\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right] \\ &= \frac{1}{4}(\langle\partial_u f, \partial_u f\rangle - \langle\partial_v f, \partial_v f\rangle - 2i\,\langle\partial_u f, \partial_v f\rangle) \\ &= \frac{1}{4}(E - G - 2iF).\end{aligned}$$

Then X is isothermal $\iff E = G, F = 0 \iff \phi^2 = 0$.

The second part is easy to show. By Corollary 19, it is sufficient to show x_k is harmonic $\iff$ φ_k is holomorphic. With Lemma 20, we see that

$$\partial_{uu}x_k + \partial_{vv}x_k = 4\left(\frac{\partial}{\partial\bar{z}}\left(\frac{\partial f}{\partial z}\right)\right) = 4\left(\frac{\partial}{\partial\bar{z}}(\varphi_k)\right).$$

By Definitions 13 and 15, x_k is harmonic and φ_k is holomorphic simultaneously if and only if the above expression equal to zero.

□

Now we are very close to establishing the representation formulae. We will need to solve $\varphi_k = \frac{\partial x_k}{\partial z}$ for x_k to get a formula for constructing minimal surfaces by finding suitable holomorphic functions φ_k.

Lemma 22. *Let $\phi = (\varphi_1, \varphi_2, \varphi_3)$ with φ_k being holomorphic functions such that*

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0 \qquad \text{and} \qquad |\phi|^2 \neq 0 \text{ and is finite,}$$

we have the parametrisation

$$X = \left(Re\int\varphi_1(z)dz, Re\int\varphi_2(z)dz, Re\int\varphi_3(z)dz\right)$$

representing a minimal surface.

Proof. We first look at the constraints we have. If we have $\phi^2 = 0$ then X is isothermal

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c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

and the following

$$\begin{aligned}
|\phi|^2 &= \left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \left|\frac{\partial x_3}{\partial z}\right|^2 \\
&= \frac{1}{4}\left[\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2 + \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2\right] \\
&= \frac{1}{4}(\langle \partial_u f, \partial_u f\rangle + \langle \partial_v f, \partial_v f\rangle) \\
&= \frac{1}{4}(E+G) \\
&= \frac{E}{2}
\end{aligned}$$

shows we must have $|\phi|^2 \neq 0$ for these surfaces to exist.

We solve $\varphi_k = \frac{\partial x_k}{\partial z}$ for x_k. Notice that x_k is a function of two variables z and $\bar{z}$, but we want to have a representation with respect to one variable. This can be achieved using differentials [6]. We first observe that

$$\begin{aligned}
dx_k &= \frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv \quad \text{and} \\
dz &= du + idv.
\end{aligned} \tag{6}$$

Using the Wirtinger derivatives, we have

$$\begin{aligned}
\varphi_k dz = \frac{\partial x_k}{\partial z}dz &= \frac{1}{2}\left(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v}\right)(du + idv) \\
&= \frac{1}{2}\left[\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv + i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right], \\
\overline{\varphi_k dz} = \overline{\frac{\partial x_k}{\partial z}}\,\overline{dz} &= \frac{1}{2}\left(\frac{\partial x_k}{\partial u} + i\frac{\partial x_k}{\partial v}\right)(du - idv) \\
&= \frac{1}{2}\left[\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv - i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right].
\end{aligned}$$

Adding these gives

$$\varphi_k dz + \overline{\varphi_k dz} = \frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv = 2Re\{\varphi_k dz\}. \tag{7}$$

Combining (6) and (7), we have

$$dx_k = 2Re\{\varphi_k dz\}.$$

Postal Address: 111 Barry Street
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Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
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Hence by Cauchy's integral formula, $x_k = 2Re\int \varphi_k dz + C_k$. Since the constants C_k and 2 are just translating and scaling factors and do not change the geometric shape of the surface, we can simply discard them and leave the coordinate function to be

$$x_k = Re\int \varphi_k dz$$

as desired. □

Following Lemma 22, there is only one step left to construct the Weierstrass Representations i.e., determining φ_k. As one may guess, there are different approaches to this and there are also different forms of Weierstrass Representation [5]. Here we will only introduce one most common form having two complex functions as parameters.

Theorem 23 (Weierstrass Representation(p,q))**.** *[2] Every regular minimal surface has locally an isothermal parametrisation of the form $f(z)$ equal to*

$$\left(\mathrm{Re}\left\{\int^z p(1+q^2)dw\right\}, \mathrm{Re}\left\{\int^z -ip(1-q^2)dw\right\}, \mathrm{Re}\left\{\int^z -2ipq\, dw\right\}\right)$$

in some domain $D \subseteq \mathbb{C}$, where p is holomorphic and q is meromorphic in D, with p vanishing at the poles of q and having a zero of order at least $2m$ wherever q has a pole of order m.

Proof. The conditions placed on p and q ensure that p, pq^2, and pq are holomorphic, therefore φ_k are holomorphic. We compute

$$\begin{aligned}\phi^2 &= \left[p(1+q^2)\right]^2 + \left[-ip(1-q^2)\right]^2 + [-2ipq]^2\\ &= \left[p^2 + 2p^2q^2 + p^2q^4\right] - \left[p^2 - 2p^2q^2 + p^2q^4\right] - \left[4p^2q^2\right]\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}|\varphi|^2 &= \left|p(1+q^2)\right|^2 + \left|-ip(1-q^2)\right|^2 + \left|-2ipq\right|^2\\ &= |p|^2\left[(1+q^2)(1+\bar{q}^2) + (1-q^2)(1-\bar{q}^2) + 4q\bar{q}\right]\\ &= 4|p|(1+|q|^2) \neq 0.\end{aligned}$$

We conclude that the chosen functions φ_k $(k = 1, 2, 3)$ satisfy Lemma 22 as required. □

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c/- The University of Melbourne
Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

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