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True But Not Provable

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1 Introduction

Gödel's *Incompleteness Theorem*, proven in 1931, shows that if a theory capable of expressing elementary arithmetic with recursively enumerable axioms is consistent, then a statement can be formed in that theory that is true, but cannot be proven so. The proof of Gödel's Theorem is constructive: that is to say if one starts with a set of axioms that fits the conditions, then the proof leads to a statement that is true but not provable under those axioms. The Peano axioms are an example of such a theory, and there is hence a readily available example of a statement that is true but not provable under Peano arithmetic. However, Gödel's proof is heavily logical, and leads to a statement that is not particularly natural.

The unprovable nature of Gödel's statement arises in part from self-reference: the statement can essentially be thought of as the phrase "This statement cannot be proven". During the 20th century some accessible examples were found of statements that can be expressed in the language of a basic system, are provably true using higher-order techniques, but are not provable in the basic system. Moreover, these properties are not self-referential in nature.

The unprovability of the theorems examined in this paper arises from their extraordinarily large growth. In fact, a standard method of asserting unprovability is to show that numbers involved with the properties grow too fast to be defined in that system. For example, the Goodstein Sequence for 2 and 3 reach 0 after 3 and 6 steps respectively. However, the Goodstein Sequence for 4 reaches 0 after an astounding $3 \times 2^{402,653,211} - 3$ steps [2]. The length of the Goodstein Sequence for 5 is greater than the fourth Ackermann function at 4. Furthermore, the length of Friedman's long finite sequences involving k characters, denoted $n(k)$, begins with $n(1) = 3$, $n(2) = 11$, but $n(3)$ becomes incomprehensibly large to the point that it is known to be greater

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than the 7,198th Ackermann function at 158,386 [3]. His TREE function, describing a sequence involving trees, grows even faster, with TREE(3) $> n(4)$ [10].

This paper will first give the necessary background information to prove four unprovable theorems, with significant focus on ordinals. The sections afterwards will be dedicated to proving the theorems. Due to time constraints, the unprovability of the theorems will not be explored in-depth. However, some comments will be given.

2 Preliminaries

2.1 Peano Axioms

The *Peano Axioms*, sometimes called the *Dedekind-Peano Axioms* were first introduced by Richard Dedekind in 1888 as a set of axioms to describe the natural numbers. They were later presented more precisely by Giuseppe Peano in 1889. Since then, they have remained nearly completely unchanged in their use in metamathematical investigations. One such application is in questions of completeness and consistency of number theory. The Peano Axioms are as follows.

Definition 2.1 *A set* $\mathbb{N}$ *along with a function* $S : \mathbb{N} \mapsto \mathbb{N}$ *(called the* successor function*) is said to be a* set of natural numbers *if the following conditions hold:*

1. $0 \in \mathbb{N}$
2. *If* $n \in \mathbb{N}$ *then* $S(n) \in \mathbb{N}$
3. $S(n) \neq 0$ *for all* $n \in \mathbb{N}$
4. *For all* $n, m \in \mathbb{N}$, *if* $S(n) = S(m)$ *then* $n = m$.
5. *If 0 has property* P *and for every* $n \in \mathbb{N}$, *and the condition "*$S(n)$ *has property* P *provided* n *has property* P*" holds, then for every* $n \in \mathbb{N}$, n *has property* P.

The fifth axiom is known as the *induction axiom*. For this paper, we will have $0 = \varnothing$ and $S(n) = n \cup \{n\}$. This gives $1 = S(0) = \varnothing \cup \{\varnothing\} = \{\varnothing\}$, $2 = S(1) = 1 \cup \{1\} = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$, $3 = 2 \cup \{2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ and so on. For ease of notation, this will be written as $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$ and so on. In this sense, a natural number can be thought of as the set of all natural numbers that it is greater than. In fact, it will be proven now.

Lemma 2.2 *If* $n \in \mathbb{N}$ *then* $n = \{m \in \mathbb{N} : m < n\}$.

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Proof: Firstly, $1 = \varnothing \cup \{\varnothing\} = \{\varnothing\} = \{0\}$. So $1 = \{m \in \mathbb{N} : m < 1\}$. Now assume that the result holds for some $n \in \mathbb{N}$. Then $n + 1 = n \cup \{n\}$. Since $\forall m < n$, $m \in n$, it follows that $\forall m < n$, $m \in n + 1$. Also, by definition, $n \in n + 1$. Additionally, let $m \in n+1 = n \cup \{n\}$. So either $m = n$ or $m \in n$, i.e. $m \leq n$. Hence, by induction, the result holds.

2.2 Ordinals

The *ordinal numbers* are an extension of the natural numbers that are used to identify well-ordered sets. Two ordered sets are said to have the same *order type* if and only if they are order-isomorphic. Any well-ordered set is isomorphic to an ordinal number, and as such, the ordinal numbers are used to describe the order types of well-ordered sets. Their applications are fruitful, showing significant use in proof theory, and in this paper are used in Section 3 to prove that any Goodstein Sequence terminates and in Section 1 to show that any strategy against a hydra is a winning strategy.

Definition 2.3 *A non-empty set A along with a relation $\leq$ is said to be an* ordered set *if $\leq$ is reflexive, transitive and antisymmetric. That is, for all $a, b, c \in X$*

1. $a \leq a$,
2. $a \leq b$ *and* $b \leq c \implies a \leq c$,
3. $a \leq b$ *and* $b \leq a \implies a = b$.

Definition 2.4 *An ordered set A is a* chain *if and only if for all $a, b \in A$, either $a \leq b$ or $b \leq a$.*

Definition 2.5 *If an ordered set A is a chain then A is said to be* well-ordered *if and only if every non-empty subset of A has a minimum.*

Definition 2.6 *A well-ordered set α is defined to be an* ordinal number *if it has the following properties:*

1. *α is well-ordered*
2. *If $\beta \in \alpha$ then $\beta \subseteq \alpha$*

The natural numbers defined by $0 = \varnothing$ and $S(n) = n \cup \{n\}$ can be shown to be ordinals. Condition (1) holds since a natural number is a finite chain. Condition (2) holds following Lemma 2.2. Additionally, the set of natural numbers is an ordinal, with its order type denoted ω.

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Definition 2.7 *The order relation on the ordinals is defined as follows. For two ordinals* α *and* β, $\alpha < \beta$ *if and only if* $\alpha \in \beta$. *Also,* $\alpha \leq \beta$ *if and only if* $\alpha < \beta$ *or* $\alpha = \beta$.

Remark 2.8 *Strictly speaking, this is not a relation on every ordinal, since the totality of all ordinals cannot form a set. Formally, the ordinals form a class. Briefly, if* Ω *is the set of all ordinals, then* Ω *is an ordinal as well. Then* $\Omega + 1$ *(with addition to be defined next) is an ordinal as well, so* $\Omega + 1 \in \Omega$. *Then* $\Omega < \Omega + 1$ *and* $\Omega + 1 < \Omega$, *a contradiction. As a relation applies to a set, the ordering on ordinals inevitably leaves out some ordinals. However, since the relation can be applied to an arbitrary set of ordinals, no issues are encountered.*

Theorem 2.9 *Any set of ordinals is well-ordered by* $<$.

*Proof:*Let A be a set of ordinals. Let $\alpha \in A$ and let $B = \{\gamma \in A : \gamma < \alpha\}$. If B is empty, then α is the minimal element of A. Assume B is non-empty. Since α is well-ordered and $B \subseteq \alpha$, let β be the minimum of B. Then β is also the minimum of A. Hence, any set of ordinals has a minimum, and thus any set of ordinals is well-ordered.

As the ordinal numbers are an extension of the natural numbers, it is to be expected that standard arithmetical operations of addition, multiplication and exponentiation are also defined.

Definition 2.10 *Let* α, β *be ordinal numbers. The* sum $\alpha + \beta$ *of* α *and* β *is defined as the order type of the set* $(\{0\} \times \alpha) \cup (\{1\} \times \beta)$ *ordered by the following rule:*

$$(i, \xi) \leq (j, \zeta) \iff i < j \text{ or } (i = j \text{ and } \xi \leq \zeta)$$

Definition 2.11 *Let* α, β *be ordinal numbers. The* product $\alpha \cdot \beta$ *of* α *and* β *is the order type of the Cartesian product* $\beta \times \alpha$ *ordered by the following rule:*

$$(\xi, \zeta) \leq (\mu, \eta) \iff \xi < \mu \text{ or } (\xi = \mu \text{ and } \zeta \leq \eta)$$

Note that ordinal addition and multiplication is not necessarily commutative. For example, $1 + \omega = \omega \neq \omega + 1$ and $2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega$. In fact, for any two ordinals α, β, if $\alpha < \beta$, then $\alpha + \beta = \beta$.

The next definition is required to define ordinal exponentiation.

Definition 2.12 *An ordinal* α *is said to be a* successor ordinal *if and only if there is another ordinal number* β *such that* $\beta + 1 = \alpha$. *An ordinal* α *is said to be a* limit ordinal *if and only if it is not a successor ordinal.*

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Definition 2.13 *Let* α *and* β *be ordinal numbers. Then* ordinal exponentiation *is defined as follows:*

If $\beta = 0$ *then* $\alpha^\beta = 1$*.*

If β *is a successor ordinal, choose* γ *such that* $\beta = \gamma + 1$*. Then* $\alpha^\beta = (\alpha^\gamma) \cdot \alpha$*.*

If β *is a limit ordinal, and* $\alpha = 0$ *then* $\alpha^\beta = 0$*. If* $\alpha \neq 0$ *then* α^β *is the supremum of the set* $\{\alpha^\gamma : \gamma < \beta\}$*.*

Next, an extension of mathematical induction from natural numbers to any well-ordered set is given.

Theorem 2.14 (Principle of Transfinite Induction [6]) *Let* A *be a well-ordered set and let* $B \subseteq A$*. If the following condition holds for any* $x \in A$

$$\{y \in A : y < x\} \subseteq B \implies x \in B$$

then $B = A$*.*

Proof: Assume that $A \setminus B \neq \varnothing$ and assume that the inductive condition given above holds. Let $x = \min(A \setminus B)$. As x is minimal, no elements of $\{y \in A : y < x\}$ are in $A \setminus B$. So $\{y \in A : y < x\} \subseteq B$, and hence $x \in B$, which contradicts $x \in A \setminus B$. Hence, $A \setminus B = \varnothing$, i.e. $B = A$.

This allows induction over any well-ordered set and is adapted to an inductive argument by defining the set $B \subseteq A$ to be the set of elements for which a property P is true.

Finally, in order to present the main result of this section, the following two results are stated without proof, adapted from the website ProofWiki (`http://www.proofwiki.org`).

Theorem 2.15 (Division Theorem For Ordinals [7]) *Let* α *and* β *be ordinals with* $\beta \neq 0$*. Then there exist unique ordinals* γ *and* μ *such that* $\alpha = (\beta \cdot \gamma) + \mu$ *where* $\mu < \beta$*.*

The Division Theorem For Ordinals is essentially an ordinal version of Euclid's Division Algorithm for integers.

Lemma 2.16 (Unique Ordinal Exponentiation Inequality [8]) *Let* α *and* β *be ordinals with* $\alpha > 1$ *and* $\beta > 0$*. Then there exists a unique ordinal* γ *such that* $\alpha^\gamma \leq \beta < \alpha^{\gamma+1}$*.*

Once again, the Unique Ordinal Exponentiation Inequality is a generalisation to the ordinals of the statement that if n and x are natural numbers with $n > 1$, then n will lie between two successive powers of x.

Definition 2.17 *If α is an ordinal and can be written in the following form*

$$\sum_{i=1}^{k} \omega^{a_i} \cdot b_i$$

where $k \in \mathbb{N}$, $a_1, a_2, \ldots, a_i$ is a strictly decreasing sequence of ordinals, and each $b_i \in \mathbb{N}$, then this form is said to be the Cantor Normal Form *of α.*

This concludes the machinery required to derive the main result of this section, as follows. The proof is adapted from ProofWiki [9].

Theorem 2.18 *Any ordinal α can be uniquely written in Cantor Normal Form.*

Proof: Let α be an ordinal with $\alpha > 0$. Assume that $\forall \xi < \alpha$ there exists a unique Cantor Normal Form of ξ.

Since $\omega > 1$, by the Unique Ordinal Exponential Inequality, there exists a unique ordinal γ such that

$$\omega^\gamma \leq \alpha < \omega^{\gamma+1}.$$

Now by The Division Theorem For Ordinals, there exists unique ordinals β, ν where $\nu < \omega^\gamma$ such that

$$\alpha = \omega^\gamma \cdot \beta + \nu$$

Since $\alpha < \omega^{\gamma+1}$, it follows that $\beta < \omega$. Now if $\nu = 0$, the statement holds true with $\alpha = \omega^\gamma \cdot \beta$. Assume that $\nu \neq 0$. As stated above, $\nu < \omega^\gamma$ so by the Unique Ordinal Exponential Inequality, $\nu < \alpha$. Hence, by the inductive hypothesis, there is a unique Cantor Normal Form for ν, i.e.

$$\nu = \sum_{i=1}^{n} \omega^{c_i} \cdot d_i$$

for some $n, d_i < \omega$ and some strictly decreasing sequence of ordinals $c_1, c_2, \ldots, c_n$. Hence,

$$\alpha = \omega^\gamma \cdot \beta + \sum_{i=1}^{n} \omega^{c_i} \cdot d_i.$$

Then set $a_1 = \gamma$, $a_{i+1} = c_i$ and $b_1 = \beta$, $b_{i+1} = d_i$. Then

$$\alpha = \sum_{i=1}^{n+1} \omega^{a_i} \cdot b_i.$$

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Also, since γ, β and ν are unique, as well as the Cantor Normal Form for ν, it follows that the Cantor Normal Form for α is unique.

Finally, since the Cantor Normal Form for a natural number k is simply $\omega^0 \cdot k$, the result holds by transfinite induction.

Two ordinals in Cantor Normal Form can be compared by iteratively comparing each term in succession. At the first difference, the ordinal which has the larger component is the larger ordinal.

2.3 Trees

This section defines some basic properties of trees, which are used for Paris and Kirby's Hydra Game, and for Friedman's Internal Finite Tree Embeddings. It is concluded with a proof of König's Tree Lemma, which is used in proofs throughout the paper.

Definition 2.19 *An ordered set T is called a* (rooted) tree *if there is a minimum element of T, called the* root node, *and for each $x \in T$, the set $\{y : y \leq x\}$ forms a chain. If T is finite, then it is called a* finite tree.

Definition 2.20 *Let T be a tree and let $x, y \in T$ and assume $x < y$. Then y is called a* descendant *of x. If there exists no $z \in T$ such that $x < z < y$, then x is said to be the* parent node *of y and y is said to be a* child node of x.

Definition 2.21 *Let T be a tree and let $x \in T$. A* branch *of x is the set $\{y \in T : y \geq x\}$.*

Definition 2.22 *Let T be a tree and let $x \in T$. The* valence *of x is defined to be the number of child nodes of x. If a node has valence 0, then it is said to be a* top node. *The valence of T is defined to be the supremum of the valences of all its nodes. If the valence of T is a natural number, then T is called* finitely branching.

Definition 2.23 *Let T be a tree and let $x \in T$. The* height *of x is the number of elements in the set $\{y \in T : y < x\}$. The height of T is the supremum of the heights of every $x \in T$.*

Theorem 2.24 (König's tree lemma) *Let T be an infinite, finitely branching tree. Then T has a branch of infinite length.*

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Proof: Let T be a tree. It will be shown that there is a sequence of nodes $t_1, t_2, \ldots$ such that t_1 is the root node, each t_{n+1} is a child of t_n, and each t_n has an infinite number of descendants. Then the sequence $t_1, t_2, \ldots$ is the desired branch.

Let t_1 be the root node of T. Suppose all children of t_1 had a finite number of descendants. Then t_1 would also have a finite number of descendants, which would mean T is finite. Hence, there is a child of t_1 with infinite descendants. Let t_2 be any one of these children.

Now, suppose t_n has infinitely many descendants. By the same argument as above, t_n has at least one child node with infinitely many descendants. Thus, let t_{k+1} be any one of these nodes. Then the result holds by induction.

3 Goodstein's Sequence

The first example involves a sequence first described by Goodstein [1]. Goodstein was a *finitist*, meaning he believed in the philosophy of finitism, wherein a mathematical object does not exist unless it can be constructed from the natural numbers in a finite number of steps. In his paper, he made use of a more general version of the Goodstein sequence to prove the *restricted ordinal theorem* under his philosophy, which asserts that the set of ordinals less than ϵ_0 is well-ordered.

However, due to his requirement for a constructive argument, Goodstein's own proof of termination is somewhat complicated. A simpler proof can be given that requires only two conditions: that the ordinal numbers are well-ordered, and that a corresponding ordinal sequence for a given Goodstein sequence is decreasing. Theorem 2.9 showed that any set of ordinals is well-ordered. The other condition will be shown here.

The form of the sequence given in this section was described by Kirby & Paris [2]. In the same paper, it was shown that the sentence asserting that a given Goodstein sequence terminates is not provable in first-order Peano Arithmetic.

Definition 3.1 *Let $m, n \in \mathbb{N}$ with $n > 1$. Define the* hereditary base n representation *of m recursively as follows: if $m \leq n$ write m as m. Otherwise, write m as a sum of powers of n, with the exponents in hereditary base n representation.*

For example, if $m = 25$ and $n = 2$, write $25 = 2^4 + 2^3 + 1 = 2^{2^2} + 2^{2+1} + 1$. Or if $m = 45955$ and $n = 3$, first write $45955 = 3^9 \cdot 2 + 3^8 + 3^3 + 1$ and then $45955 = 3^{3^2} \cdot 2 + 3^{3 \cdot 2+2} + 3^3 + 1$.

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Definition 3.2 *Let $m, n \in \mathbb{N}$ with $n > 1$. Define the number $G_n(m)$ as follows. If $m = 0$ then $G_n(m) = 0$. Otherwise, write m in its hereditary base n representation. Then $G_n(m)$ is the result of replacing every n by $n+1$ and then subtracting 1.*

For example, $G_2(25) = 3^{3^3} + 3^{3+1}$ and $G_3(45955) = 4^{4^2} \cdot 2 + 4^{4\cdot 2+2} + 4^4$.

Definition 3.3 *Let $m \in \mathbb{N}$. Define the* Goodstein Sequence for m *by $m_0 = m$, $m_k = G_{k+1}(m_{k-1})$ for any $k > 1$.*

The Goodstein Sequence for $m = 2$ or $m = 3$ reach 0 rather quickly. For example,

$$\begin{array}{ll} 2_0 = 2 & 3_0 = 2^1 + 1 \\ 2_1 = 3^1 - 1 = 2 & 3_1 = 3^1 + 1 - 1 = 3^1 \\ 2_2 = 2 - 1 = 1 & 3_2 = 4^1 - 1 = 3 \\ 2_3 = 1 - 1 = 0 & 3_3 = 3 - 1 = 2 \\ & 3_4 = 2 - 1 = 1 \\ & 3_5 = 1 - 1 = 0 \end{array}$$

However, for larger numbers, the Goodstein Sequence grows very rapidly. For example,

$$\begin{aligned} 25_0 &= 25 = 2^{2^2} + 2^{2+1} + 1 \\ 25_1 &= 3^{3^3} + 3^{3+1} = 7625597485068 \\ 25_2 &= 4^{4^4} + 4^4 \cdot 3 + 4^3 \cdot 3 + 4^2 \cdot 3 + 4^1 \cdot 3 + 3 \approx 10^{154} \\ 25_3 &= 5^{5^5} + 5^5 \cdot 3 + 5^3 \cdot 3 + 5^2 \cdot 3 + 5^1 \cdot 3 + 2 \approx 10^{2184} \end{aligned}$$

Next, in order to show that the Goodstein Sequence for m terminates, it is shown that there is a decreasing sequence of ordinals in correspondence with the original sequence.

Definition 3.4 *Let $m, n \in \mathbb{N}$ with $n > 1$. Define the ordinal $o_n(m)$ as the result of replacing every n with ω in the hereditary base n representation of m.*

Note that $o_n(m) = 0$ if and only if $m = 0$.

Lemma 3.5 *Let $m \in \mathbb{N}$. For any Goodstein Sequence, $(m_0, m_1, m_2, \ldots) = (m, G_2(m), G_3(m_1), \ldots)$, there is a corresponding sequence of ordinals, $o_2(m), o_3(m_1), o_4(m_2), \ldots$ which is strictly decreasing.*

Proof: Suppose $n, \ell \in \mathbb{N}$ and $n > 1$, with

$$\ell = n^k \cdot a_k + n^{k-1} \cdot a_{k-1} + \ldots + n \cdot a_1 + a_0$$

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and each $a_i < n$.

For $x \in \mathbb{N}$ or $x = \omega$, let

$$f^{\ell,n}(x) = \sum_{i=0}^{k} x^{f^{i,n}(x)} \cdot a_i,$$

with $f^{0,n}(x) = 0$. This function gives the result of writing ℓ in hereditary base n form and then replacing each n with x. Note that $f^{\ell,n}(n+1) = G_n(\ell)+1$ and $f^{\ell,n}(\omega) = o_n(\ell)$. Firstly,

$$o_n(\ell) = \sum_{i=0}^{k} \omega^{f^{i,n}(\omega)} \cdot a_i,$$

and

$$G_n(\ell) = \left(\sum_{i=0}^{k} (n+1)^{f^{i,n}(n+1)} \cdot a_i \right) - 1.$$

Let $j \leq k$ be minimal such that $a_j \neq 0$. If $j = 0$ then

$$G_n(\ell) = \left(\sum_{i=1}^{k} (n+1)^{f^{i,n}(n+1)} \cdot a_i \right) + (a_0 - 1).$$

And so

$$o_{n+1}(G_n(\ell)) = \left(\sum_{i=1}^{k} \omega^{f^{i,n}(\omega)} \cdot a_i \right) + (a_0 - 1),$$

then clearly, $o_n(\ell) > o_{n+1}(G_n(\ell))$.

Now assume $j \neq 0$. Then

$$\begin{aligned} G_n(\ell) =& \left(\sum_{i=j+1}^{k} (n+1)^{f^{i,n}(n+1)} \cdot a_i \right) + (n+1)^{f^{j,n}(n+1)} \cdot (a_j - 1) \\ &+ \sum_{i=0}^{j-1} (n+1)^{f^{i,n}(n+1)} \cdot n. \end{aligned}$$

So

$$o_{n+1}(G_n(\ell)) = \left(\sum_{i=j+1}^{k} \omega^{f^{i,n}(w)} \cdot a_i \right) + \omega^{f^{j,n}(\omega)} \cdot (a_j - 1) + \sum_{i=0}^{j-1} \omega^{f^{i,n}(\omega)} \cdot n.$$

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Both $o_{n+1}(G_n(\ell))$ and $o_n(\ell)$ agree up until the jth term. The jth term in $o_{n+1}(G_n(\ell))$ is $\omega^{f^{j,n}(\omega)} \cdot (a_j - 1)$, and the jth term in $o_n(\ell)$ is $\omega^{f^{j,n}(\omega)} \cdot a_j$. Since $a_j > (a_j - 1)$, it follows that $o_n(\ell) > o_{n+1}(G_n(\ell))$.

Hence, the sequence $o_2(m), o_3(m_1), o_4(m_2), \ldots$ is strictly decreasing.

Theorem 3.6 (Goodstein's Theorem) *For any $m \in \mathbb{N}$, the Goodstein Sequence for m eventually reaches zero.*

Proof: By Lemma 3.5, for any Goodstein Sequence $m, m_1, m_2, \ldots$, there is a corresponding sequence of strictly decreasing ordinals. By Theorem 2.9, any set of ordinals is well-ordered and hence this sequence is finite, i.e., there exists $n \in \mathbb{N}$ where $o_{n+2}(m_n) = 0$. Hence, $m_n = 0$ and thus the Goodstein Sequence for m terminates.

4 The Hydra Game

The *Hydra Game* is visualised as a battle between Hercules and a *hydra*, where a hydra is simply any finite tree. A *head* of a given hydra is any of its top nodes. Hercules wishes to defeat the hydra by cutting off one head at a time, however, the hydra is capable of growing new heads. A battle between Hercules and a hydra proceeds as follows: at stage $n \geq 1$ in the game, Hercules selects one head on the hydra to cut off. Let a be the node that was cut and let b be the parent node of a. Firstly, remove a from the tree. Then, if b is the root node, no new heads are grown. Otherwise, from the parent of b, attach n copies of the branch of b after the cut. Figure 1 shows an example of the first 3 stages of a battle.

A *strategy* for Hercules is a function that determines which head to cut from the hydra at each stage. For some strategy σ, σ is a *winning strategy* if and only if it eventually reduces the hydra to just the root node. As it turns out, every strategy is a winning strategy, the proof of which will begin now.

Firstly, assign to each node v an ordinal $f(v)$ defined as follows. Let $C(v)$ denote the set of children of v. If v is a top node, then $f(v) = 0$. Otherwise, $f(v) = g(C(v))$ where g is defined as follows.

For a set X of nodes in a tree, order the set X such that for $v, w \in X$, $v < w$ iff $f(v) < f(w)$. Now label every element of X such that

$$v_1 \geq v_2 \geq \cdots \geq v_k.$$

Then $g(X) = \omega^{f(v_1)} + \omega^{f(v_2)} + \cdots + \omega^{f(v_k)}$.

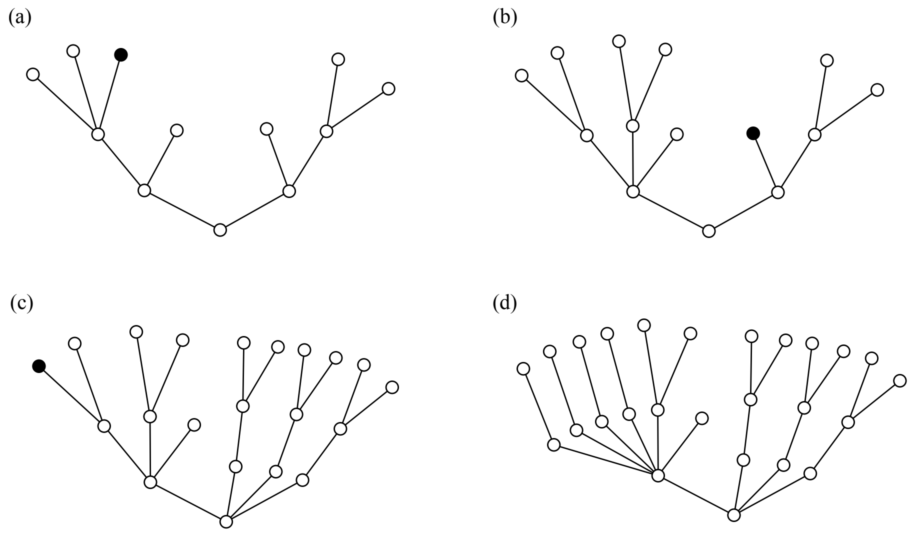

Figure 1: In each image, the shaded node is the node that is cut off in the next stage. (a) before stage 1, (b) after stage 1, (c) after stage 2, (d) after stage 3.

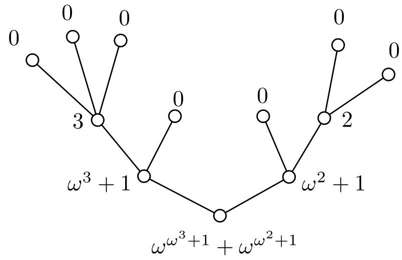

Figure 2: The ordinal assignment for each node in the hydra from Figure 1(a)

The ordinal assigned to a given hydra is the ordinal that is assigned to its root node. An example of the ordinal assignment for the hydra in Figure 1(a) is shown in Figure 2.

Lemma 4.1 *For any strategy, the ordinal for the hydra at stage n will be greater than the ordinal at stage $n+1$.*

Proof: Consider the game at the stage $n-1$. Let c be the node that is cut, let b be the node that is replicated and let a be the node that the new nodes grow from. For any node v, let v' denote the same node after the cut.

After the cut, the children of node b' will be $C(b) \setminus \{c\}$. The children of node a' will be $(C(a)\setminus b)\cup\{n \text{ copies of } b'\}$ (i.e., the branch for b' that remains as well as $n-1$ extra copies). The label for node a' will now be $f(a') = g((C(a) \setminus \{b\}) \cup \{n \text{ copies of } b'\})$.

Now, as the children of b' will be identical to the children of b with the exception of node c, it follows that

$$f(b') = g(C(b')) = g(C(b) \setminus \{c\}) < g(C(b)) = f(b).$$

Now, consider the ordinal label for a,

$$f(a) = \omega^{b_1} \cdot a_1 + \omega^{b_2} \cdot a_2 + \ldots + \omega^{b_k} \cdot a_k$$

where each $a_i \in \mathbb{N}$.

Assume that $b_j = f(b)$. Since $f(b') < f(b)$ it follows that

$$f(a') = \omega^{b_1} \cdot a_1 + \ldots + \omega^{b_j} \cdot (a_j - 1) + \ldots + \omega^{b_\ell} \cdot (a_l + n) + \ldots + \omega^{b_k} \cdot a_k$$

where $b_\ell = f(b')$.

If $a_j > 1$ then comparing $f(a')$ and $f(a)$ will have the first term difference between $\omega^{b_j} \cdot (a_j - 1)$ and $\omega^{b_j} \cdot a_j$. Then clearly, $f(a') < f(a)$.

If $a_j = 1$ then the first term difference will be between $\omega^{b_{j+1}} \cdot a_{j+1}$ and $\omega^{b_j} \cdot a_j$. By defintion, $b_j > b_{j+1}$ and hence $f(a') < f(a)$.

Similarly, for any node v where a is a descendant of v, it will follow that $f(v') < f(v)$. Since a descends from the root node r, it follows that $f(r') < f(r)$.

This concludes the requirements to prove the main result for this section.

Theorem 4.2 *Any strategy is a winning strategy.*

Proof: Let r_i denote the root node of the ordinal at stage i for a given strategy. By Lemma 4.1, the sequence of ordinals $f(r_1), f(r_2), f(r_3), \ldots$ is a strictly decreasing sequence. By Theorem 2.9, this sequence is well-ordered and hence is finite. Thus, there exists $n \in \mathbb{N}$ such that $f(r_n) = 0$. So, by definition, r_n is a top node and hence Hercules wins.

The statement "Every strategy is a winning strategy" was shown by Kirby and Paris to be unprovable in Peano arithmetic by constructing a specific strategy which cannot be proven to be winning [2].

5 Long Finite Sequences

In this section, a certain property of sequences will be observed, described by Friedman [3]. It will be shown that for any finite alphabet, there is a longest finite sequence in

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that alphabet for which no consecutive block $x_i, x_{i+1}, \ldots, x_{2i}$ is a subsequence of a later block $x_j, x_{j+1}, \ldots, x_{2j}$. Allowing $n(k)$ to denote the longest possible length of such a sequence, it will be seen that $n(1) = 3$, $n(2) = 11$, and that $n(3)$ is incomprehensibly large. The theorem that states there is a longest sequence for which the property holds cannot be proven using two-quantifier logic. It requires, at minimum, three alternating quantifiers.

For any set A, let A^* denote the set of finite sequences from A, including the empty sequence. Also, for any sequence x, let $x[i]$ denote the i-th term of x.

Definition 5.1 *Let $k \geq 1$ and let $x, y \in \{1, \ldots, k\}^*$. The sequence x is said to be a* subsequence *of y if and only if there exist $1 \leq i_1 < i_2 < \cdots < i_n \leq m$ such that for all $j \leq n$, $x[j] = y[i_j]$. The definition also holds for infinite sequences x and y by omitting "$< i_n \leq m$", and for finite sequence x and infinite sequence y by omitting "$\leq m$".*

Definition 5.2 *Let $k \geq 1$ and let $x \in \{1, \ldots, k\}$. The sequence x has the* Friedman Property *if and only if there does not exist $i < j \leq n/2$ such that $x[i], x[i+1], \ldots, x[2i]$ is a subsequence of $x[j], x[j+1], \ldots, x[2j]$. An infinite sequence $x[1], x[2], x[3], \ldots$ has the Friedman Property if and only if there does not exist $i < j$ such that $x[i], x[i+1], \ldots, x[2i]$ is a subsequence of $x[j], x[j+1], \ldots, x[2j]$.*

If it exists, define $n(k)$ to be the length of the longest finite sequence in $\{1, \ldots, k\}^*$ with the Friedman Property.

Theorem 5.3 *Let $k \geq 1$ and let x be an infinite sequence from $\{1, \ldots, k\}$. Then x does not have the Friedman Property. Moreover, let $y_1, y_2, \ldots$ be elements of $\{1, \ldots, k\}^*$. Then there exists $i < j$ such that y_i is a subsequence of y_j.*

Proof: First we show that the second claim implies the first claim. Let $x[1], x[2], \ldots$ be elements of $\{1, \ldots, k\}$. Define $y_i = (x[i], x[i+1], \ldots, x[2i])$. Then according to the second claim, there exists $i < j$ such that y_i is a subsequence of y_j. Then the sequence $x[1], x[2], \ldots$ does not have the Friedman Property.

Now assume that the second claim is false. Define a sequence of sequences $y_1, y_2, \ldots$ where each $y_i \in \{1, \ldots, k\}^*$ as a *bad sequence* if and only if it is a counterexample to the second claim.

Next, a minimal bad sequence is constructed in a technique first used by Nash–Williams [5] which Friedman made use of in his paper. Let $y_1 \in \{1, \ldots, k\}^*$ be of minimal length such that y_1 starts some bad sequence. Let $y_2 \in \{1, \ldots, k\}^*$ be of minimal length such that y_1, y_2 starts some bad sequence. Continue defining elements in such a way such that $y_1, y_2, y_3, \ldots$ is a bad sequence.

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Now choose an infinite subsequence of y_i's such that the first term is the same for each of them. Denote this list $\bar{y}$. Let n be defined such that $y_n = \bar{y}[1]$. Now let the sequence z be defined such that each z_i is equal to $\bar{y}[i]$ with the first term omitted.

Now, the sequence z is also a bad sequence. For if it were not, then there would exist $i < j$ such that z_i is a subsequence of z_j. But then $\bar{y}[i]$ would be a subsequence of $\bar{y}[j]$, so $y_1, y_2, y_3, \dots$ would not be a bad sequence, contradicting its definition.

Similarly, the sequence $y_1, y_2, \dots, y_{n-1}, z_1, z_2, \dots$ is also bad. But z_1 is shorter than y_n, violating the definition of y_n as minimal. Hence, the second claim is true.

Theorem 5.4 *Let $k \geq 1$. Then there is a longest sequence from $\{1, \dots, k\}^*$ that has the Friedman Property.*

Proof: Consider the tree T of all elements of $\{1, \dots, k\}^*$ with the Friedman Property, under extension (e.g. 122 is an extension of 12 which is an extension of 1) and rooted by the empty sequence. Then T is a finitely branching tree. If T has infinitely many nodes, then by König's Tree Lemma 2.24, T has an infinite path. However, this infinite path results in an infinite sequence with the Friedman Property, contradictory to Theorem 5.3. Hence, T has finitely many nodes.

Noting that if $x, y \in \{1, \dots, k\}^*$, x has the Friedman Property, and x extends y, then y also has the Friedman Property. Hence, the height of the tree T will be the length of the maximal sequences. That is, the height of T is $n(k)$.

The next result is also due to Friedman. They will be stated without proof here but can be found in Friedman's paper [3]. The value for $n(2)$ is simply a case-by-case analysis of what a sequence can start with, and the result for $n(1)$ is clear.

Theorem 5.5 $n(1) = 3$. $n(2) = 11$.

The longest sequence with the Friedman Property in $\{1\}^*$ is 111, and the longest sequences in $\{1, 2\}^*$ are 12221111111 and 21112222222. However, there does not seem to even be a glimpse of hope that one could write down the longest sequence with the Friedman Property in $\{1, 2, 3\}^*$, for reasons which will soon become clear. Before this result is presented, a version of the Ackermann hierarchy of functions is defined.

Definition 5.6 *Define the function $A_k(n) : \mathbb{N} \mapsto \mathbb{N}$ as follows. $A_1(n) = 2n$. $A_{k+1}(n) = A_k(A_k(A_k(\cdots A_k(1) \cdots)))$ where there are n copies of A_k.*

For purposes of illustration, some calculations from [3] are given.

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$A_3(1) = 2$. $A_3(2) = 4$. $A_3(3) = 16$. $A_3(4) = 2^{16} = 65,536$. $A_3(5) = 2^{65,536}$
$A_4(1) = 2$
$A_4(2) = A_3(A_3(1)) = A_3(2) = 4$
$A_4(3) = A_3(A_4(2)) = A_3(4) = 2^{16}$
$A_4(4) = A_3(A_4(3)) = A_3(65,536) = 2^{2^{\cdot^{\cdot^{2}}}}$ where there are 65,536 copies of 2.

It is fair to say that $A_4(4)$ is a very large number, but it is not quite incomprehensible—one could picture a tower of 2's with a large height. However, $A_4(5)$ is a tower of 2's of height $A_4(4)$ which is arguably incomprehensible. One could imagine that $A_5(5)$ is even larger, with Friedman proposing it as a sort of "benchmark" for incomprehensibly large numbers. It is for these reasons that the following result is so remarkable.

Theorem 5.7 (Friedman's Theorem 6.9 [3]) $n(3) > A_{7198}(158386)$.

It is the large growth of $n(k)$ that results in its unprovable nature. The function $n(k)$ eventually dominates all provably recursive functions of 2-quantifier arithmetic, but is a provably recursive function for 3-quantifier arithmetic. Hence, the statement "For all k, $n(k)$ exists" is provable in 3-quantifier arithmetic but not 2-quantifier arithmetic.

6 Internal Finite Tree Embeddings

Kruskal's Tree Theorem, due to Joseph Kruskal, is a theorem asserting that for any infinite sequence of trees, there is a tree that can be embedded into a later tree. Later, Friedman showed that there are special cases of the theorem that cannot be proven in first-order arithmetic [4]. Here, Friedman's special case, called the *Internal Finite Tree Embedding Theorem*, will be proven. In order to do so, firstly, some definitions are given.

Definition 6.1 *For a finite tree T and $x, y \in T$, the* inf *operation is defined such that $x \inf y$ is the greatest $z \in T$ such that $z \leq x$ and $z \leq y$.*

Let T_1, T_2 be finite trees. The function h is said to be an inf preserving embedding *from T_1 into T_2 if and only if the following conditions hold:*

1. *$h : T_1 \mapsto T_2$ is one-to one, and*
2. *for all $x, y \in T_1$, $h(x \inf y) = h(x) \inf h(y)$.*

Definition 6.2 *Define an n-*labelled tree*, where $n \geq 1$, as a tree T along with a function $\sigma : T \mapsto \{1, \ldots, n\}$. The function σ is called the* label function for T.

Let T_1, T_2 be n-labelled trees with $n \geq 1$ and respective labellings σ_1, σ_2. A function $h : T_1 \mapsto T_2$ is said to be label preserving if and only if for all $x \in T_1$, $\sigma_1(x) = \sigma_2(h(x))$.

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Definition 6.3 *Let T_1, T_2 be finite trees. The function $h : T_1 \mapsto T_2$ is defined to be* terminal preserving *if and only if for all $x \in T_1$, if x is a top node in T_1, then $h(x)$ is a top node in T_2.*

Kruskal's Tree Theorem, which will not be proven here, is as follows. The statement of the theorem is adopted from Friedman [4], and a simple proof was developed by Nash-Williams [5].

Theorem 6.4 (Kruskal's Tree Theorem) *Let $T_1, T_2, \ldots$ be an infinite sequence of finite trees. Then there exist $i, j \in \mathbb{N}$ such that $i < j$ and there is an inf-preserving embedding from T_i into T_j.*

The following theorem is given by Friedman as a special case of Kruskal's Tree Theorem.

Theorem 6.5 *[4] Let $n \geq 1$ and $T_1, T_2, \ldots$ be an infinite sequence of finite n-labelled trees. Then there exists $i, j \in \mathbb{N}$ such that $i < j$ and there is an inf and label preserving embedding from T_i into T_j.*

This allows the following theorem to be derived.

Theorem 6.6 *Let $n \geq 1$ and $T_1, T_2, \ldots$ be an infinite sequence of finite n-labelled trees. Then there exists $i, j \in \mathbb{N}$ such that $i < j$ and there is an inf, label and terminal preserving embedding from T_i into T_j.*

Proof: Let $T'_1, T'_2, \ldots$ be $2n$-labelled trees derived from $T_1, T_2, \ldots$ where each node of valence 0 has n added to its label, and every other node remains the same. Then the sequence $T'_1, T'_2, \ldots$ meets the conditions for Theorem 6.5, and hence there exists $i < j$ such that there exists an inf and label preserving embedding from T'_i into T'_j. This will preserve the labels of each terminal node, and since only terminal nodes have label $> n$, it follows that this embedding will be an inf, label and terminal preserving embedding from T_i into T_j.

This allows the derivation of the Internal Finite Tree Embedding Theorem. Firstly, define a *full tree* as a tree where all nodes of valence 0 have the same height, and all nodes of valence > 0 have the same valence. Define a *truncation* of a tree T to be the tree obtained by restricting T to all nodes whose height is as most some given natural number.

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Theorem 6.7 (Internal Finite Tree Embedding Theorem) *Let $k, n \geq 1$ and let T be a sufficiently tall full finite n-labelled tree of valence k. Then there is an inf, label and terminal preserving embedding from some truncation of T into some truncation of T of greater height.*

Proof: Let $k, n \geq 1$ and suppose the theorem is false. The full finite n-labelled trees of valence k or valence 0 form an infinite finitely branching tree TR under the relation of truncation.

Let TR$'$ be the full finite n-labelled trees of valence k or 0 for which the theorem holds false, under the relation of truncation. Then TR$'$ will also be an infinite finitely branching tree. Hence, by König's Tree Lemma 2.24, TR$'$ contains an infinite path.

This infinite path will form a sequence of trees $T_1, T_2, \dots$ of full finite n-labelled trees of valence k or 0 for which the theorem fails, and where each T_i is the truncation of height i of T_{i+1}. However, by Theorem 6.6, there exists $i, j \in \mathbb{N}$ such that $i < j$ and there is an inf, label and terminal preserving embedding from T_i into T_j. Hence, the Internal Finite Tree Embedding Theorem is true for T_j, which is a contradiction.

The following theorem is similarly unprovable in first-order arithmetic, and is in fact shown by Friedman to be equivalent to The Internal Finite Tree Embedding Theorem.

Theorem 6.8 *For all $k, n \geq 1$ there exists r such that the following holds. Let $T_1, T_2, \dots, T_r$ be full n-labelled trees of valence k or 0, where for each $1 \leq i \leq r$, T_i has height i. Then there exists $i, j \in \mathbb{N}$ such that $i < j \leq r$ with an inf and label preserving embedding from T_i into T_j.*

Proof: Suppose the theorem is false. Then there exists $k, n \geq 1$ such that for all $r \in \mathbb{N}$, every finite sequence $T_1, T_2, \dots, T_r$ of full n-labelled trees of valence k or 0 where each T_i has height i has no $i < j \leq r$ such that there is an inf and label preserving embedding from T_i into T_j.

Let TR be the set of trees such that the above holds under the relation of truncation. Then TR forms an infinite finitely branching tree, and hence by König's Tree Lemma 2.24, TR contains an infinite path. This infinite path forms an infinite sequence $T_1, T_2, \dots$. By Theorem 6.5, there exists $i < j$ such that there is an inf and label preserving embedding from T_i into T_j. Hence, with $r = j$, Theorem 6.8 is true, a contradiction.

As a final example of the extraordinarily large growth associated with these unprovable theorems, the next result is given. Friedman [10] showed this by using a sequence of size $n(4)$ (defined in Section 5) to construct a sequence of trees that meets the conditions of Theorem 6.8. Firstly, let TREE(n) be the length of the longest sequence $T_1, T_2, \dots, T_k$ of trees that meet the condition for Theorem 6.8.

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Theorem 6.9 TREE(3) $> n(4)$.

7 Conclusion

One key factor in all of these theorems is the extremely large growth rate associated with them. Even the Ackermann function, known as one of the fastest growing total computable functions, is often used to define lower bounds for the growth rates involved. Friedman, understandably, seems to take great pleasure in his incomprehensibly large numbers, describing Graham's number as "puny" in size compared to $n(4)$ and TREE(3) on FOM, a foundations of mathematics e-mail list [11].

The Goodstein sequence presented in this paper was the sequence that resulted from increasing the base by 1 at each step in the sequence. The sequence originally described by Goodstein was as follows. Let σ_n be a non-decreasing sequence. Rather than first writing the number in hereditary base 2 representation then increasing the base by 1 at each step, begin the sequence with the hereditary base σ_1 representation and after the n-th step, increase the base to hereditary base σ_{n+1} [1].

Let $m \in \mathbb{N}$ and let σ_n be a non-decreasing sequence. Using the notation described in Lemma 3.5, let m_n be the sequence such that where $m_1 = m$ and $m_n = f^{m_{n-1},\sigma_{n-1}}(\sigma_n) - 1$. Goodstein's paper showed that the sequence m_n eventually terminates.

Now let $h_1^\sigma(m) = m$ and $h_n^\sigma(m) = f^{h_{n-1}^\sigma(m),\sigma_{n-1}}(\sigma_n) - 1$. For instance, if $\sigma_n = n$ then $h_n^\sigma(m)$ is simply the standard Goodstein sequence.

Then, let $g_1(m)$ be the length of the sequence $h_n^n(m)$. Naturally, this forms an infinite sequence $g_1(1), g_1(2), \ldots$. Let g_1 denote this sequence. Next, for $k \in \mathbb{N}$ define the sequence g_k such that $g_k(m)$ is the length of the sequence $h_n^{g_{k-1}}(m)$. Assuming that $g_k(m+1) \geq g_k(m)$, the sequence g_{k+1} eventually terminates. As an example, the sequence $g_2(m)$ is the length of the sequence obtained by starting at m, and at stage i, increasing the base by the length of the Goodstein sequence for i.

Finally, let $\mathcal{G}(m)$ be the length of the sequence $h_n^\sigma(m)$ where $\sigma_i = g_i(i)$. Assuming that $g_i(i) \leq g_{i+1}(i+1)$, then this sequence also terminates. No investigations have been done in this direction, but, due to each $\mathcal{G}(m)$ being defined in terms of other incomprehensibly large numbers, it seems that the growth of the function $\mathcal{G}(m)$ would be *incomprehensibly incomprehensible*. One could continue to define Goodstein sequences in such a way, forming a hierarchy of Goodstein sequences.

Also, it was mentioned in Section 2.2 that the ordinal numbers show significant use in proof theory. For instance, in proof theory, ordinal analysis assigns an ordinal number to a theory as a measure of its strength. The *proof theoretic ordinal* is the smallest recursive ordinal which cannot be proven to be well-ordered. For instance,

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the proof theoretic ordinal of Peano arithmetic is ϵ_0, where ϵ_0 is the smallest ordinal ϵ satisfying the equation $\epsilon = \omega^\epsilon$. It can also be shown that $\epsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$.

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