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K-graphs and non-commutative geometry.

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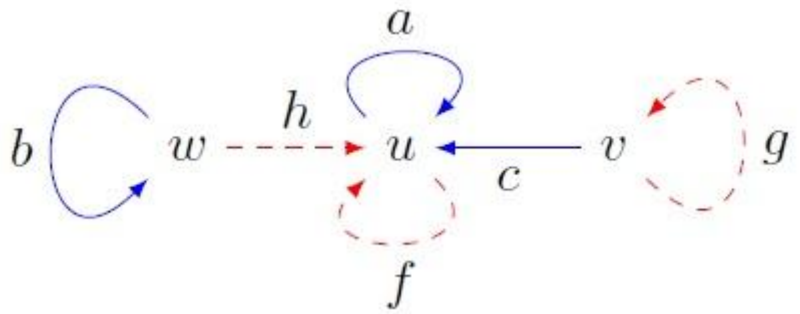
Originally described in 2000 by Kumjian and Pask, k-graphs are a 'higher-dimensional graph' – technically they are described as a countable category Λ with a distance functor $(d\colon \Lambda\colon \mathbb{N}^k)$ satisfying a *factorization* requirement – for all λ such that $d(\lambda) = m + n$ there are μ, ν such that $d(\mu) = m, d(\nu) = n$, and $\lambda = \mu\nu$. They can be studied pictorially via their *1-skeleton*, which is a graph with the edges coloured. Here we identify the vertices with each functor of degree zero, and edges with functors of minimal non-zero degree.

The *Cuntz-Krieger construction* (which is similar to the construction from a directed graph) lets us use a k-graph to generate a C*-algebra in a natural way by associating generating elements with each vertex and edge in the k-graph. Many important qualities of the created algebra can be determined from the relations inside the graph (these would otherwise be difficult or impossible to calculate).

Examples of these are the homology, cohomology and K-theory of a given k-graph (and associated Cuntz-Krieger algebra). Homology and cohomology are dual theories, one attempting to give a sense of the holes in a space at different dimensionalities, and the other attempting to do something similar to the dual space (space of functions on the k-graph). K-theory gives an idea of the dimensions of a space (in this case the C*-algebra), by measuring the projections upon it. These quantities allow us to understand what the algebra looks like.

In this study, we attempted to characterise an example of a k-graph and associated Cuntz-Krieger algebras for k>2. The k-graph we used was generalised from the following example given by Kumjian, Pask and Sims:



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It was realized that the kth graph of this type starts by emitting edges from a single point to vertices below it, which then gained loops (edges e with s(e)=r(e)) based on the colours of vertices pointing to them. This allows for a fairly natural construction using sets – given the set of numbers $N_k = \{1 \dots k\}$, we associate vertices with subsets of this set subject to a saturation requirement (which only allows a vertex to exist if all vertices below it already do), and we let path of every distance $n \in \mathbb{N}^k$ be emitted from every vertex. We then call this graph Λ_W where W is the set containing each vertex.

The homology of this k-graph can then be found by considering the relative homology as described by Hatcher – this gives an excision formula, which in turn shows that $H_n(\Lambda_W) = H_n(\mathbb{T}^k) = \mathbb{Z}^{\binom{k}{n}}$ (for $n < k$), where $\mathbb{T}$ is the circle in the complex plane. We can then invoke the Universal Coefficient Theorem of Kumjian, Pask and Sims to show that $H^n(\Lambda_W, \mathbb{T}) = \mathbb{T}^{\binom{k}{n}}$. The K-theory is found by invoking the Pimsner-Voiculescu 6-term exact sequence, which immediately gives us that $K_0(\Lambda_W) = K_1(\Lambda_W) = \mathbb{Z}$.

Next, we invoked the 2-cohomology of our k-graph to show that there existed a set of 2-cocycles $e_i + e_j \longmapsto e^{2\pi i \theta_{ij}}$. We can define elements T_i as the sum of partial isometries associated with edges of a single colour i, and under the twisting from these 2-cocycles, these elements fulfill the generalised quantum n-sphere relations, which were first studied by Baum, Hajac, Matthes and Szymanski for the quantum 3-sphere:

$$\prod_{i \in N_k} (1 - T_i T_i^*) = 0$$
$$T_i^* T_i = 1$$
$$T_i T_j = e^{2i\pi\theta_{ij}} T_j T_i$$
$$T_i T_j^* = e^{-2i\pi\theta_{ij}} T_j^* T_i$$

This example has prompted interest in the idea of k-graph relative homology, which has never been described before. In particular, we hope to give a class of k-graph quotients with zero homology, which in turn allows for excision of these classes whilst retaining the homology of the k-graph – implying that the k-graph and the sub-k-graph that give these homologies are homotopic to some extent.

Furthermore, as we are describing a non-commutative object here, we hope to be able to create a modular spectral triple – a spectral triple itself cannot be defined using the Cuntz-Krieger algebra of our example because we cannot define a faithful graph trace, however there is hope we can identify a KMS-state on it. This has been done before for a graph by Carey, Marcolli and Rennie (among others possibly), but to the best of our knowledge it has not been done for a k-graph, which has interesting differences to a normal graph.

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References

Kumjian, A., and Pask, D., 2000, 'Higher Rank Graph C*-Algebras', *New York Journal of Mathematics*, vol. 6, pp. 1-20

Kumjian, A., Pask, D., and Sims, A., 'Homology for Higher Rank Graphs and Twisted C*-algebras', *In Preparation.*

Baum, P., Hajac, P., Matthes, R., and Szymanski, W., 2005, 'The K-theory of Heegard-type Quantum 3-spheres', *K-Theory*, vol 35, pp. 159-186

Carey, A., Marcolli, M., and Rennie, A., 'Modular Index Invariants of Mumford Curves', *In Preparation.*