INTERNATIONAL CENTRE OF EXCELLENCE FOR EDUCATION IN MATHEMATICS

Representations of the General Linear Algebra on Polynomial Rings over C

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I have been looking at a representation of the general linear algebra of gl_n and $gl_n + gl_m$ on polynomial rings over C using differential operators. These representations are completely reducible. Moreover, these rings can be written as a direct sum of irreducible submodules in which each submodule appears only once.

gl_n on $C[x_1,\dots x_n]$

Let E_{ij} be the standard basis for gl_n, which implies $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$. Define a linear map φ: $gl_n \rightarrow$ $End(C[x_1,\dots x_n])$ by $\varphi(E_{ij}) = x_i \frac{\partial}{\partial x_j}$. In order to see that this is a representation, and hence that $C[x_1,\dots x_n]$ (which we'll often write as C[*X*]) is a gl_n-module, we need to show that φ preserves the bracket, i.e. $\varphi([E_{ij}, E_{kl}]) = [\varphi(E_{ij}),\varphi(E_{kl})]$. A simple calculation shows that this is the case, and thus φ is an algebra representation, and C[X] a gl_n module.

Irreducible submodules of C[X]

The action of an element E_{ij} in gl_n permutes an element *f*(X*)* of C[X] in a predictable way. If a term in *f*(X*)* contains x_j, for example, E_{ij} swaps one copy of that term with *xi.* If a term in *f*(X*)* doesn't contain x_j, it is reduced to 0.

E.g. $E_{12}(x_1 x_2 x_3 + x_1) = (x_1 x_2 x_3 + x_1) = (x_1)^2 x_3$

Notice that while the action of gl_n permutes indices of terms in a polynomial, it doesn't change the degree of a term, unless it kills the term off entirely. So the spaces of 'homogeneous polynomials' (eg. <{$x_1, x_2, \dots, x_n$}>, <{$(x_1)^2, x_1 x_2, \dots, (x_n)^2$ }>, etc) are subspaces of C[X] which are invariant under the action of gl_n — they are gl_n-submodules. We will call the space of homogeneous polynomials of degree k 'V_k'. These spaces are irreducible (they have no non-trivial submodules) since gl_n acts transitively on V_k. Thus

$$\mathbb{C}[x_1, ..., x_n] = \bigoplus_{k\geq 0} V_k$$

It turns out that the space V_k can be constructed by a linear combinations of a single vector $(x_1)^k$ and its image under the action of $\{E_{ij} \mid i > j\}$. This vector, which also satisfies various other properties, is known as the 'highest weight vector' for the space V_k.

$(gl_n + gl_m)$ - modules

We can also make the ring $C[x_1^1, x_1^2, \dots, x_1^m, \dots, x_n^1, \dots, x_n^m]$ into a $(gl_n + gl_m)$ module using differential operators. Again the spaces of homogeneous polynomials, V_p, are submodules of C[X], but this time they're not irreducible. Highest weight vectors for the irreducible spaces are products of Δ_k, where $k \leq min\{n, m\}$, and

$$\triangle_k = det \begin{pmatrix} x_1^1 & \dots & x_1^k \\ \vdots & \ddots & \vdots \\ x_k^1 & \dots & x_k^k \end{pmatrix}.$$

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See http://www.ice-em.org.au/students.html#scholarships2007