Symmetric groups, symmetric functions and quantum groups

Jonathan Sun, School of Mathematics and Statistics, University of Sydney

Over the summer of 2005/06, I undertook a vacation scholarship under Prof. Gus Lehrer at the University of Sydney. Overall I found it to be a very rewarding experience. The scholarship was not only challenging but I was also able learn a lot about research maths. The topics I looked at were symmetric functions, the classical Hall algebra and representation of quivers. I will give the basic definitions to give some flavour of each topic.

Symmetric Functions. The notion of a symmetric polynomial is familiar from high school, where the coefficients of a monic quadratic polynomial, x^2+bx+c, can be written as a symmetric polynomial in the roots x_1, x_2. Namely, $b=-x_1-x_2$ and $c=x_1x_2$.

For any integer n, we have the ring of polynomials in n variables with integer coefficients, $\mathbb{Z}[x_1,\dots,x_n]$. The symmetric group S_n acts on this ring by permuting the variables. A *symmetric polynomial* is a polynomial that is fixed under this action. The set of all symmetric polynomials forms a subring $\Lambda_n=\mathbb{Z}[x_1,\dots,x_n]^{S_n}$.

We would like to study these symmetric polynomials for arbitrary large n. The *ring of symmetric functions* Λ is the generalisation to countably ínfinite number of variables. Formally this is described using a inverse limit. I looked at several bases for the ring of symmetric functions. These all turned out to be indexed by the set of all partitions, where a partition is a sequence of non-negative integers in weakly decreasing order and only finitely many terms are non-zero.

Classical Hall Algebra. Fix an arbitrary prime p. By a finite p-group we mean a group with order p^α for some integer α. Let $H(p)$ be the free $\mathbb{Z}$-module with basis the set of equivalence classes of finite abelian p-groups. Any finite abelian p-group is isomorphic to a direct product of cyclic group of the form $\mathbb{Z}/p^n\mathbb{Z}$. So If G is a finite abelian p-group then $G\simeq\prod_i\mathbb{Z}/p^{\lambda_i}\mathbb{Z}$, where the λ_i are weakly decreasing. That is, a equivalence classes of finite abelian p-groups is uniquely determined by a partition. Let $\lambda=(\lambda_1,\lambda_2,\dots)$ be a partition, define $G_\lambda(p)=\prod_i\mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ and $u_\lambda(p)$ to be the basis element in $H(p)$ corresponding to the equivalence class of $G_\lambda(p)$. So $H(p)$ has basis $\{u_\lambda(p)|\ \lambda$ a partition $\}$.

We define a product on $H(p)$, $u_\lambda(p)u_\mu(p)=\sum_\nu H^\nu_{\lambda\mu}(p)u_\nu(p)$, where $H^\nu_{\lambda\mu}(p)=\mathrm{Card}\{N\le G_\nu|\,N\simeq G_\lambda,\,G_\nu/N\simeq G_\mu\}$.

$H(p)$ is called the *classical Hall algebra*. We can prove that $H^\nu_{\lambda\mu}(p)$ is a polynomial in p that is independent of the choice of p. So we can define a generic classical Hall algebra where the multiplication constants are replaced by the correponding polynomial. I looked an alternative way of describing the classical Hall algebra which involved $Gl_n(k)$, k a finite field.

Representation of Quivers. A quiver is just a set of points with arrows between them, loops and multiple arrows are allowed. More formally, a *quiver* Q is a pair (X,A) where X is a finite set, and A is a collection of ordered pairs (h,t), $h,t\in X$. A k-representation of a quiver Q assigns a k-vectorspace $V(x)$ to each point $x\in X$, and a linear map $\phi(a)$ to each arrow $a=(h,t)\in A$, $\phi(a):V(h)\to V(t)$. Here k is any field. Note that this is not a commutative diagram, so suppose we have the quiver $X=\{x,y,z\}$, $A=\{a=(x,y),b=(y,x),c=(x,z)\}$, it is not necessarily true that $\phi(c)=\phi(b)\phi(a)$. For each quiver Q, we can define a k-algebra, the path algebra kQ. The path algebra has basis the set of paths, where a path is a finite sequence of points $(x_1,\dots,x_n)$ and arrows $a_i=(x_i,x_{i+1})$ connecting them. The multiplication on kQ is concatenation if the end point of the first path is also the start point of the second, and zero otherwise. We can show that a k-representation of Q is equivalent to a kQ-module. This is reminiscent of the equivalence of representations of a finite group and modules over the group algebra. I looked at some basic quivers and found the irreducible and indecomposable representations.