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Group Actions on the Cohomology of Hyperplane Complements

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This project has studied the topological space M obtained from $\mathbb{C}^l$ by removing a finite set of hyperplanes $\mathcal{A}$. Associated to this space is its cohomology ring, which contains information about the topology of the space. When there is a symmetry group G acting on $\mathcal{A}$, it also acts on the cohomology ring $H^*(M) = \bigoplus_{n\in\mathbb{N}} H^n(M)$ and we study the representation $T : G \to GL(H^*(M))$. An important special case is the space of configurations $M_l = \{(z_1, \cdots, z_l) \in \mathbb{C}^l : z_i \neq z_j \text{ if } i \neq j\}$ with $G = S_l$.

Cohomology

Let $M \subset \mathbb{C}^l$ be a smooth manifold. One way to compute the cohomology spaces $H^n(M, \mathbb{C})$ is to use the vector spaces $\Omega^n(M)$ of holomorphic n-differential forms on M. These are expressions of the form $\omega = \sum f_{i_1,\cdots,i_n} dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ where $1 \leq i_1 < \cdots < i_n \leq l$ and $f_{i_1,\cdots,i_n}$ are holomorphic functions from M to $\mathbb{C}$. The wedge $\wedge$ is a product on these forms, and has the property that $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

Definition 0.1 *The exterior derivative d acts on n-forms and outputs n+1-forms by the following rule: If ω is as above, then*

$$d\omega = \sum_{j=1}^{l} \sum \frac{\partial f_{i_1,\cdots,i_n}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

Let d_n be the restriction of the exterior derivative to the n-forms. The de Rham complex of M is the following sequence of vector spaces and maps:

$$0 \to \Omega^0(M) \overset{d_0}{\to} \Omega^1(M) \overset{d_1}{\to} \Omega^2(M) \overset{d_2}{\to} \Omega^3(M) \overset{d_3}{\to} \cdots$$

An important property of the exterior derivative is that $\forall n$, $(d_{n+1} \circ d_n)(\omega) = 0$ *for any differential n-form* ω. *This implies that the image of* d_n *is contained in the kernel of* d_{n+1}, *so the quotient* $\ker d_{n+1} / \operatorname{im} d_n$ *is well defined. We define the n-th cohomology group of* M *to be* $H^n(M) = \ker d_n / \operatorname{im} d_{n-1}$ *and the cohomology group of* M *to be* $H^*(M) = \bigoplus_{n \in \mathbb{N}} H^n(M)$. H^* *inherits a ring structure from* $\Omega^*(M) = \bigoplus_{p \in \mathbb{N}} \Omega^p(M)$.

Hyperplane Complements

Definition 0.2 *Suppose* V *is a vector space of dimenson* l *over a field* k. *A hyperplane* H *in* V *is a vector subspace of dimension* $l-1$. *An arrangement* $\mathcal{A}$ *is a finite set of hyperplanes in* V.

Example 0.3 *Considering* $\mathbb{R}^3$ *as a real vector space, a hyperplane is simply a plane through the origin.*

Definition 0.4 *If* $\mathcal{A}$ *is an arrangement of hyperplanes, then* $M_{\mathcal{A}} = V \setminus \bigcup_{H \in \mathcal{A}} H$ *is said to be a hyperplane complement. A special example is the arrangement* $\mathcal{A} = \{H_{ij} : x_i - x_j = 0\}$ *which yields the hyperplane complement* $M_{\mathcal{A}} = \{(x_1, \cdots, x_l) : x_i \neq x_j \text{ if } i \neq j\} \subseteq \mathbb{C}^l$. *This space is called a configuration space.*

From this point we will always take the underlying field to be $k = \mathbb{C}$. Orlik and Solomon were able to obtain a generators and relations description of the cohomology ring $H^*(M_{\mathcal{A}})$ when A is a complex arrangement. For a hyperplane $H \in \mathcal{A}$ we define the 1-form $\omega_H = \frac{dL_H}{L_H}$ where L_H is a linear form such that $\ker L_H = H$.

Example 0.5 *If* H *is given by the equation* $x_1 - x_2 = 0$ *then*

$$\omega_H = \frac{d(x_1 - x_2)}{x_1 - x_2} = \frac{1}{x_1 - x_2} dx_1 - \frac{1}{x_1 - x_2} dx_2.$$

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Theorem 0.6 (Orlik-Solomon) *$H^*(M_{\mathcal{A}})$ is generated as an associative algebra by $\{[\omega_H] : H \in \mathcal{A}\}$ where $[\omega_H]$ is the image of ω_H under the quotient map used to define the cohomology groups. All the relations these generators satisfy may be deduced from:*

1)

$$\omega_H \wedge \omega_{H'} = -\omega_{H'} \wedge \omega_H$$

2) If $H_1, \cdots, H_k$ are hyperplanes such that $L_1, \cdots, L_k$ are linearly dependent (so that the codimension of $\cap H_i$ is less than k) then

$$\sum_{i=1}^{k} (-1)^i \omega_{H_1} \wedge \cdots \wedge \widehat{\omega_{H_i}} \wedge \cdots \wedge \omega_{H_k} = 0$$

where the hat denotes omission of that term.

Example 0.7 *If H_{ij} denotes the hyperplane with equation $x_i - x_j = 0$, then H_{12}, H_{23}, H_{13} are linearly dependent since $(x_1 - x_2) + (x_2 - x_3) - (x_1 - x_3) = 0$. So then we have $-\omega_{H_{23}} \wedge \omega_{H_{13}} + \omega_{H_{12}} \wedge \omega_{H_{13}} - \omega_{H_{12}} \wedge \omega_{H_{23}} = 0$.*

Recall the configuration space $M_l = M_{\mathcal{A}} \subseteq \mathbb{C}^l$ where $\mathcal{A} = \{H_{ij} : z_i - z_j = 0\}$. The symmetric group on l letters, S_l, acts on M_l by permutation of coordinates, and this action transfers to an action on the cohomology ring $H^*(M_l)$.
The action of $\pi \in S_l$ on $\omega_{ij} = \omega_{H_{ij}}$ is given by the rule

$$\pi\omega_{ij} = \omega_{\pi i, \pi j}.$$

Example 0.8 *M_3 is the space obtained by removing the planes*

$$H_{12} : z_1 - z_2 = 0, H_{23} : z_2 - z_3 = 0, H_{13} : z_1 - z_3 = 0$$

from $\mathbb{C}^3$. The cycle $(13) \in S_3$ acts on a point $(z_1, z_2, z_3) \in M_3$ to produce (z_3, z_2, z_1). By the previous theorem, the cohomology ring $H^*(M_3)$ is generated by the forms $\omega_{12} = \frac{d(z_1 - z_2)}{z_1 - z_2}, \omega_{23} = \frac{d(z_2 - z_3)}{z_2 - z_3}$ and $\omega_{13} = \frac{d(z_1 - z_3)}{z_1 - z_3}$. S_3 acts on the cohomology ring. E.g. The cycle $\pi = (132)$ acts on the first generator as such: $\pi\omega_{12} = \omega_{31}$.

In 1987, G.I. Lehrer computed trace$(g, H^p(M_n))$ for $g \in S_n$ - that is, the trace of the linearized action of each element of S_n on the p-th cohomology group of M_n.

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Remark 0.9 *The trace is invariant under conjugacy, and each conjugacy class of the symmetric group is determined by the cycle types. So up to conjugacy,* $g \sim l_1^{n_1} l_2^{n_2} \cdots l_r^{n_r}$. *Since* g *has* n_i *cycles of length* l_i *we have the condition that* $\sum n_i l_i = n$.

Theorem 0.10 (G.I. Lehrer, 1987) *: Suppose* g *has cycle type* $l_1^{n_1} l_2^{n_2} \cdots l_r^{n_r}$. *Define the Poincaré polynomial of* M_n *by*

$$P(g,t) = \sum_{p \in \mathbb{N}} \mathrm{trace}(g, H^p(M_n)) t^p.$$

Let $p_n(t) = \sum_{d|n} \mu(n/d)(-t)^{n-d}$ *where* $\mu(n)$ *is the Möbius function, which is defined to be be* $1, -1$ *if* n *is square-free with an even or odd number of prime factors respectively, and* 0 *if* n *is not square-free. E.g* $p_1(t) = 1, p_2(t) = 1 + t, p_3(t) = 1 - t^2$. *Then* $P(g,t) = P_1(t) P_2(t) \cdots P_r(t)$ *where*

$$P_i(t) = p_{l_i}(t) \left(p_{l_i}(t) - l_i(-t)^{l_i}\right) \left(p_{l_i}(t) - 2l_i(-t)^{l_i}\right) \cdots$$

$$\cdots \left(p_{l_i}(t) - (n_i - 1) l_i (-t)^{l_i}\right)$$

Corollary 0.11 *In 1969 V.I. Arnold computed the dimensions of the vector spaces* $H^p(M_n, \mathbb{C})$. *The trace of the identity is simply the dimension of the space so his result can be rephrased as*

$$P(1,t) = (1+t)(1+2t) \cdots (1 + (n-1)t).$$

Proof 0.12 *The identity on the symmetric group with* n *elements has cycle type* $(1)^n$ *- it has* n *cycles, each of length* 1. *So by the previous theorem, we have*

$$P(1,t) = P_1(t)$$

$$= p_1(t)(p_1(t) - (-t)^1)(p_1(t) - 2(-t)^1) \cdots (p_1(t) - (n-1)(-t)^1).$$

Since $p_1(t) = 1$, *this simplifies to V.I. Arnold's result.*

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