Off-Critical Logarithmic Minimal Models

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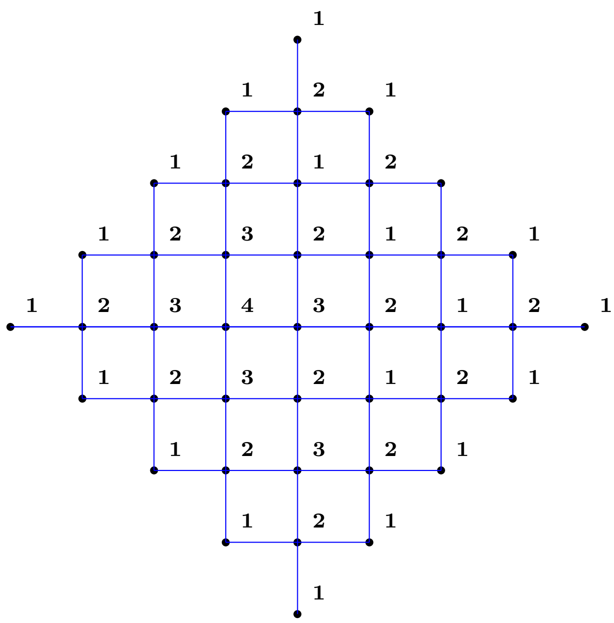
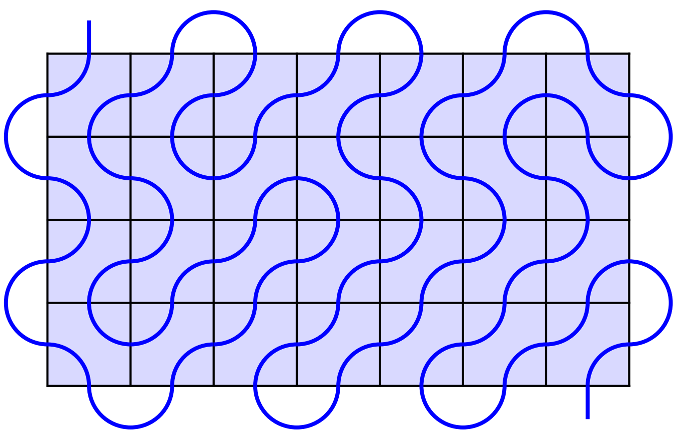
The study of statistical mechanics on 2-D lattice models has underpinned the development of a mathematical theory of phase transitions. Some special systems have free energies that can be solved for exactly (are "integrable").

The restricted solid-on-solid (RSOS) models are examples of exactly solvable lattice models (Forrester & Baxter 1985). These consist of a square lattice upon which every site is assigned an integer height, with nearest neighbour heights differing by 1 unit. In the thermodynamic limit (lattice size becomes infinite), these models include several famous examples such as the Ising model of ferromagnetism and are generally known as the minimal models, $\mathcal{M}(m, m')$ (m and m' relatively prime integrers), after the conformal field theories that they are described by in the continuum limit (Belavin, Polyakov & Zamolodchikov, 1984).



The logarithmic minimal models, $\mathcal{LM}(p, p')$ (Pearce, Rasmussen & Zuber, 2006), are integrable lattice models that, at their critical points, are described by logarithmic CFTs in the continuum limit. They can be represented as a "loop gas" in the plane, each distinguished by a different fugacity (probability of finding a loop in the gas) determined by relatively prime integers p and p'. Famous examples of these are critical dense polymers (left) and critical percolation.



Away from their critical points, these models are still integrable. By performing expansions in a temperature-like variable describing the perturbation away from criticality, the thermodynamic statistics of the lattice can be calculated.

Taking a path on the RSOS lattice from the central site to the boundary of a finite sublattice with fixed boundary heights, the heights of the sites visited constitute a walk on an A_n graph. The generating functions containing the energies of different possible walks and their degeneracies, called *1-D configurational sums*, are used in the calculation of the lattice thermodynamics. These are polynomials in the temperature-like variable and arise from the diagonalisation of the lattice corner transfer matrices (the standard technique for calculating the lattice free energy).

The RSOS models are related to the $\mathcal{LM}(p, p')$ models by the logarithmic limit - that is, the $\mathcal{LM}(p, p')$ spectral data can be approximated through a sequence of RSOS models. This limit holds for the conformal weights and Virasoro characters of the underlying CFTs - the parameters that characterise the nature of the phase transition. We used the results known for the RSOS models from Forrester and Baxter and the logarithmic limit to derive 1-D configurational sums (Y) for the $\mathcal{LM}(p, p')$ models

$$\begin{aligned} Y^{(N)}_{ab\{b+1\}}(q) &= \begin{bmatrix} N \\ \frac{N+a-b}{2} \end{bmatrix} - q^{ra} \begin{bmatrix} N \\ \frac{N-a-b}{2} \end{bmatrix} \\ Y^{(N)}_{a\{b+1\}b}(q) &= \begin{bmatrix} N \\ \frac{N+a-b-1}{2} \end{bmatrix} - q^{ra} \begin{bmatrix} N \\ \frac{N-a-b-1}{2} \end{bmatrix}, \end{aligned}$$

as well as the recursion relations that these satisfy for a truncated lattice and demonstrated that these are also robust in the logarithmic limit. Here N is the length of the walk, a the central height, b the boundary heights, $r = \lfloor \frac{bp}{p'} \rfloor + 1$ and the square brackets denote q-binomials, polynomial analogues of the binomial co-efficients. These results can then be applied to calculate statistics such as local height probabilities of the model. The 1-D configurational sums corresponding to ground state boundary conditions (corresponding to walks of locally minimal energy) also converge to the Virasoro characters as the truncation size becomes infinite.

Throughout the vacation project I learned much of the gritty, technical aspects of research, such as writing reports and presentations, while also having the opportunity to work directly upon a glamorous mathematical theory. The project was an educational test-run at research.

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References

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