AMSI Vacation Research Scholarship Project Report
Einstein metrics on domains in a solid torus.

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1 Introduction

This project dealt with the Einstein equation on an altered solid torus. The aim was to understand the fundamental concepts involved with Ricci curvature, and then solve the Dirichlet problem for the Einstein equation on this altered torus.

2 Preliminaries

Ricci Curvature

Definition 2.1 *Let $\mathcal{C}(M)$ define the set of all smooth vector fields on a smooth Riemannian manfiold, (M, G), equipped with its Levi-Civita connection. The curvature endomorphism of M is the map*

$$\mathcal{R} : \mathcal{C}(M) \times \mathcal{C}(M) \times \mathcal{C}(M) \to \mathcal{C}(M)$$

defined as

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

For all vector fields on $\mathbb{R}^n$, one can show $\mathcal{R}$ is identically zero. Thus, in a sense, $\mathcal{R}$ describes exactly how much M differs from 'flat' space [1]. $\mathcal{R}$ is a (1,3) tensor field with components $\mathcal{R}^l_{ijk}$ defined as

$$\mathcal{R}(\partial_j, \partial_k)\partial_i = \mathcal{R}^l_{ijk} \partial_l.$$

Since 4 tensors are often difficult to deal with, it is helpful to reduce the order of these tensors, while still maintaining their useful information. We first construct the Riemann curvature tensor by lowering the index of $\mathcal{R}$.

Definition 2.2 *The Riemann curvature tensor is a $(0, 4)$ tensor field with its action on vector fields defined by*

$$Rm(X, Y, Z, W) = G(R(X, Y)Z, W).$$

In coordinates it is written

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l.$$

We have the relationship

$$R_{lijk} = g_{ls} \mathcal{R}^s_{ijk}.$$

Definition 2.3 *We define the Ricci curvature of M to be the the trace of the curvature tensor, with components defined as*

$$R_{ij} = g^{lm} R_{iljm}.$$

see [4] *for more on this.*

Definition 2.4 *Let $p \in M$, and let $\sigma \subset T_p M$ be a 2 dimensional subspace of $T_p M$. Let X, Y be linearly independent vectors spanning σ. The sectional curvature of σ is defined as*

$$K(X, Y) = \frac{G(R(X, Y)X, Y)}{\|X\|^2 \|Y\|^2 - (G(X, Y))^2}.$$

One can show that the expression for K is independent of the choice of X and Y (see [2]).

Einstein Manifolds

Definition 2.5 *We say a smooth, n-dimensional Riemannian manifold manifold (M, G), is* ***Einstein*** *if it satisfies the relationship*

$$Ric(G) = \tau G, \qquad \tau \in \mathbb{R}. \tag{2.1}$$

That is, (M,G)'s Ricci curvature tensor is proportional to its Riemannian metric.

Theorem 2.6 *A smooth, 3 dimensional Riemannian manifold, (M, G), is Einstein satisfying (2.1) if and only if M has constant sectional curvature $\frac{\tau}{2}$* (see [1]).

3 The problem

When one solves the Einstein equation, one is finding a family of Riemannian metrics which are proportional to their Ricci curvature tensors. A solution to the Einstein equation on the full solid torus, $D^2 \times S^1 \subset \mathbb{R}^4$, where D^2 and S^1 are the unit disk and unit circle in $\mathbb{R}^2$ respectively, has already been found. For example, see [5] for the solution to this problem. On the full solid torus, one may employ cylindrical coordinates, (λ, μ, r), where the point (λ, μ, r) corresponds to the point $((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^4$ such that

$$x_1 = r\cos(2\pi\lambda),\ y_1 = r\sin(2\pi\lambda)$$
$$x_2 = \cos(2\pi\mu),\ y_2 = \sin(2\pi\mu).$$

When using these coordinates, it is natural to require that the equipped Riemannian metric, G, is 'rotationally symmetric'. That is, G does not depend on λ or μ. Furthermore, it's also standard to require the metric to be diagonal in these cylindrical coordinates. This means we can write the metric, G, as

$$G = f^{*2}(r) d\lambda \otimes d\lambda + g^{*2}(r) d\mu \otimes d\mu + h^{*2}(r) dr \otimes dr, \quad r \in (0, 1]. \tag{3.1}$$

We first compute the Ricci curvature of the solid torus with the metric in (3.1).

Lemma 3.1 *The Ricci curvature of the metric G given by (3.1) satisfies*

$$Ric(G) = \left(-\frac{f^* f^*_{rr}}{h^{*2}} + \frac{f^* f^*_r h^*_r}{h^{*3}} - \frac{f^* f^*_r g^*_r}{g^* h^{*2}}\right) d\lambda \otimes d\lambda$$

$$+ \left(-\frac{g^* g^*_{rr}}{h^{*2}} + \frac{g^* g^*_r h^*_r}{h^3} - \frac{g^* f^*_r g^*_r}{f^* h^{*2}}\right) d\mu \otimes d\mu$$

$$+ \left(-\frac{f^*_{rr}}{f^*} + \frac{f^*_r h^*_r}{f^* h} - \frac{g^*_{rr}}{g^*} + \frac{g^*_r h^*_r}{g^* h^*}\right) dr \otimes dr. \tag{3.2}$$

The goal of this project was to seek solutions to (2.1) on an altered torus. We introduce the notation $D^2_\varepsilon = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq \varepsilon\}$. We work on the 'hollowed out' solid torus, $\mathcal{T} := (D^2 \backslash \{D^2_\varepsilon\}) \times S^1$. We are interested in the solution of the Einstein equation on our altered manifold with boundary, and look to see if any criterion must be satisfied in order to guarantee existence of Einstein metrics. We are also able to employ the same cylindrical coordinates on $\mathcal{T}$, and thus (3.1) and (3.2) induce expressions for the metric and Ricci curvature on $\mathcal{T}$. In order to simplify calculations, we reparametrise to new coordinates $(\lambda, \mu, \tilde{r})$ such that the point $r = \varepsilon$ corresponds to the point $\tilde{r} = 0$ and $r = 1$ corresponds to $\tilde{r} = 1$. Then our metric is of the form

$$G = \tilde{f}^2 d\lambda^2 + \tilde{g}^2 d\mu^2 + \tilde{h}^2 d\tilde{r}^2.$$

As done in [5], we introduce the new parameter $s = \int_0^{\tilde{r}} \tilde{h}(\rho)\, d\rho$. Defining $\sigma = \int_0^1 \tilde{h}(\rho)\, d\rho$, the metric can now be written as

$$G = f^2(s) d\lambda^2 + g^2(s) d\mu^2 + ds^2, \quad s \in (0, \sigma].$$

We look to solve (2.1) for positive τ when the geometry on the boundaries of $\mathcal{T}$ is fixed. We set the value of G on the inner boundary of $\mathcal{T}$, call it $\partial_1 \mathcal{T}$, and the outer boundary, call it $\partial_2 \mathcal{T}$. Thus, we may write the metric on these boundaries as

$$G_{\partial_1 \mathcal{T}} = \alpha^2 d\lambda \otimes d\lambda + \gamma^2 d\mu \otimes d\mu, \tag{3.3}$$
$$G_{\partial_2 \mathcal{T}} = \beta^2 d\lambda \otimes d\lambda + \delta^2 d\mu \otimes d\mu, \tag{3.4}$$

$\alpha, \beta, \gamma, \delta > 0$. By the construction of G, the above boundary data corresponds to setting the boundary conditions on f and g

$$\begin{aligned} f(0) = \alpha, &\quad g(0) = \gamma, \\ f(\sigma) = \beta, &\quad g(\sigma) = \delta. \end{aligned} \tag{3.5}$$

Below is the main result obtained from the project:

Theorem 3.2 *Suppose $\tau > 0$, $\kappa = \sqrt{2\tau}$, $\frac{\kappa\sigma}{2} \neq n\pi$ for any $n \in \mathbb{Z}$, and that*

$$G = f^2(s) d\lambda^2 + g^2(s) d\mu^2 + ds^2, \quad s \in (0, \sigma],$$

is a rotationally symmetric Riemannian metric on $D^2 \backslash \{D^2_\varepsilon\} \times S^1$, satisfying boundary conditions (3.5). *Then G is an Einstein metric satisfying* (2.1) *if and only if*

$$\frac{\alpha\gamma + \beta\delta}{\alpha\delta + \beta\gamma} \in (0,1)$$

and

$$f(s) = \alpha \cos\left(\frac{\kappa s}{2}\right) + \frac{\beta - \alpha\cos\left(\frac{\kappa\sigma}{2}\right)}{\sin\left(\frac{\kappa\sigma}{2}\right)} \sin\left(\frac{\kappa s}{2}\right),$$

$$g(s) = \gamma \cos\left(\frac{\kappa s}{2}\right) + \frac{\delta - \gamma\cos\left(\frac{\kappa\sigma}{2}\right)}{\sin\left(\frac{\kappa\sigma}{2}\right)} \sin\left(\frac{\kappa s}{2}\right),$$

$$\sigma = \frac{2}{\kappa} \arccos\left(\frac{\alpha\gamma + \beta\delta}{\alpha\delta + \beta\gamma}\right).$$

Remark: The fact that $\frac{\kappa\sigma}{2} \neq n\pi$ ensures that the above expressions for f and g are well defined.

References

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