# Discrete Dynamics of a Bouncing Ball

Gilbert Oppy  
Dr Anja Slim  
Monash University

## Contents

1 Abstract 2

2 Introduction - Motivation for Project 2

3 Mathematical Model 3
   3.1 Mathematical Description 3
   3.2 Further Requirements of Model 4
   3.3 Non-Dimensionalization 4

4 Computational Model 5

5 Experimental Approach 6
   5.1 General Use of Model 6
   5.2 Focusing on the $\alpha = 0.99$ Case 7
   5.3 Phase Plane Analysis 8
   5.4 Analytically Finding 1-Periodic Bouncing Schemes 9

6 Phase Space Exploration 10
   6.1 A 1-Periodic regime in Phase Space 10
   6.2 Other Types of Periodic Bouncing 11
   6.3 Going Backwards in Time 13

7 Bifurcations Due to Changing the Frequency $\omega^*$ 15
   7.1 Stability Analysis for 1-Periodic Behaviour 15
   7.2 Bifurcations for the $V^* = 3$ 1-Periodic Foci 16

8 Conclusion 20

9 Acknowledgements 21
1 Abstract

The main discrete dynamics system that was considered in this project was that of a ball bouncing up and down on a periodically oscillating infinite plate. Upon exploring this one dimensional system with basic dynamical equations describing the ball and plate’s vertical motion, it was surprising to see that such a simple problem could be so rich in the data it provided. Looking in particular at when the coefficient of restitution of the system was $\alpha = 0.99$, a number of patterns and interesting phenomena were observed when the ball was dropped from different positions and/or the frequency of the plate was cycled through. Matlab code was developed to first create the scenario and later record time/duration/phase of bounces for very large number of trials; these programs generated a huge array of very interesting results. Ultimately, for different parameters, Period-1, Period-2, Period-3, Period-4, Period-5 and Period-8 phenomena were observed in the long-term motion of the ball, and it was clearly observed how 1-periodic motion was subject to period doubling as the frequency of plate oscillation was increased.

2 Introduction - Motivation for Project

Normally, when a droplet of liquid is dropped into a bath of the same fluid, the droplet quickly coalesces; that is, although it may splash initially, the liquid in the droplet becomes part of the bath. However, for some high viscosity oils, when the bath is vibrated at a suitable frequency, the droplet may instead bounce up and down on the bath’s surface. This is a result of surface tension, and the fact that when the bath is oscillating up and down at a fast enough speed, the droplet never has enough time to coalesce. Further to this, at specific frequencies and drop heights, the droplet may begin to walk across the bath’s surface (see Bush [1]). That is, the droplet will start to exhibit horizontal movement in the x-y plane to go along with the bouncing up and down. This project ultimately dealt with a discrete dynamics system that could be considered as a heavily abstracted model of this real-world phenomenon. It was hoped that the models used in this project might have the potential to be expanded upon in the future to accommodate more of the real-world physics of bouncing fluid droplets.
3 Mathematical Model

3.1 Mathematical Description

A ball bouncing up and down on a moving plate involves two distinct phases that alternately occur; when the ball is moving with parabolic trajectory in time above the oscillating plate (a ‘bounce’), and when the ball and plate collide. Simple equations of motion were utilized to describe this system. The vertical motion of the plate in time $x_{\text{plate}}(t)$, oscillating with frequency $\omega$, was to be described by the equation:

$$x_{\text{plate}} = A \sin(\omega t)$$  \hspace{1cm} (1)

The vertical motion of the ball while bouncing in between collision times was given by:

$$x_{\text{ball}} = -\frac{g}{2}(t - t_n)^2 + c(t - t_n) + d$$  \hspace{1cm} (2)

Here $t_n$ was the time at which the most recent ($n^{th}$) collision occurred, and $c$ and $d$ were the velocity and vertical position respectively that the ball had immediately upon leaving the plate at the $n^{th}$ collision. Collisions occurred at times when the positions of the ball and the plate were the same. That is, collisions occurred for roots $t \geq t_n$ of the equation:

$$f(t) = x_{\text{ball}} - x_{\text{plate}} = -\frac{g}{2}(t - t_n)^2 + c(t - t_n) + d - A \sin(\omega t) = 0$$  \hspace{1cm} (3)

After determining the time of a collision, the values for $c$ and $d$ needed to be updated to model the trajectory of the ball as it moved away from the plate on its next bounce. The equation describing the velocity of an object (in this case the ball) after a partially elastic collision with another object (in this case the plate) is given below

$$v_{\text{ball}} = \frac{\alpha m_{\text{plate}}(u_{\text{plate}} - u_{\text{ball}}) + m_{\text{ball}}u_{\text{ball}}}{m_{\text{ball}} + m_{\text{plate}}}\frac{1}{m_{\text{plate}}}$$

where $u$ and $v$ are the velocities of the objects before and after the collision respectively, and $\alpha$ is the coefficient of restitution, ranging between 0 and 1. A value of 0 for $\alpha$ indicates that the collision is perfectly inelastic, which means that all the energy in the collision will be sapped from the two object system. In the case of the ball and the plate, a value of 0 for $\alpha$ would mean that the ball would instantaneously stick to the plate at each collision, and the two would move off in unison. A value of 1 for alpha, on the other hand, indicates that the collision is perfectly elastic (no energy is lost in the collision). In short, $\alpha$ determines how well energy is conserved by the system following each collision.

Now, since the plate was to be taken as much larger than the ball, working in the limit of the plate being infinitely massive:

$$\lim_{m_{\text{plate}} \to \infty} \frac{\alpha m_{\text{plate}}(u_{\text{plate}} - u_{\text{ball}}) + m_{\text{ball}}u_{\text{ball}} + m_{\text{plate}}u_{\text{plate}}}{m_{\text{ball}} + m_{\text{plate}}}$$

then

$$v_{\text{ball}} = \frac{\alpha m_{\text{plate}}(u_{\text{plate}} - u_{\text{ball}}) + m_{\text{plate}}u_{\text{plate}}}{m_{\text{plate}}}$$

and hence

$$v_{\text{ball}} = \alpha(u_{\text{plate}} - u_{\text{ball}}) + u_{\text{plate}}$$  \hspace{1cm} (4)
Thus, after each ball-plate collision, the motion of the ball for the next bounce in (2) has $c$ updated as the new velocity of the ball from (4) and $d$ as the vertical position $x_{\text{plate}}(t)$ that the collision occurred at.

### 3.2 Further Requirements of Model

For the inelastic case where $\alpha = 0$, the ball will stick to the plate and hence move with the same velocity as the plate on each collision. When this happens, the ball can only leave the plate again when:

$$\text{accel}_{\text{plate}} < -g$$ (5)

This is because when the plate begins to accelerate downwards at more than $g$, which occurs when the plate is approaching its peak height, the plate 'drops out' from underneath the ball and the ball escapes from the plate with the speed of the plate at that instance. In these such cases, $c$ and $d$ in (2) were instead updated respectively with the velocity of the plate when (5) was first satisfied and the position in space that this first occured at. The time that this occured at was given by the smallest positive root of the equation $y(t) = 0$ below: (since looking for the first time that the acceleration of the plate falls below $g$)

$$y(t) = \text{accel}_{\text{plate}} + g = -Aw^2\sin(\omega(t - t_n)) + g = 0$$ (6)

For partially inelastic cases where $\alpha$ is low, although the ball will not immediately stick to the plate as not all the energy is sapped from the system in a collision, in many situations the size of the bounces will get smaller and smaller in a phenomenon known as chattering [3]. That is, the ball plate system will get to a state where the ball is just bouncing very rapidly with very small amplitudes and time durations on the surface of the plate. However, while in actuality this will obviously lead to the ball ultimately sticking to the plate, this situation poses problems for a numerical/computational model such as the one generated in this project. In the same vein as Zeno’s Achilles and the Tortoise paradox, a numerical model of this situation will for example have the ball bouncing in the air between collisions with the plate for $\frac{1}{2}$ a second, then $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$... and so on but never actually sticking to the plate. To combat this, it was decided that if the following condition was satisfied then the ball would be said to have stuck to the plate:

$$t_{\text{bounce}} < 0.0001s$$

That is, when the duration of bouncing was considerably small, to avoid indefinitely calculating infinitely small bounce durations and collision times a cut-off bounce time was introduced, below which the ball was defined as stuck. This ensured that the motion of the ball over a long time span could be found in finite time.

### 3.3 Non-Dimensionalization

There were a variety of parameters and variables that needed to be kept track of in this system. These were:

- Time ‘$t$’, with dimensions [T]
- Ball position (or ball height) ‘$x$’, with dimensions [L]
The amplitude of oscillation of the plate 'A', with dimensions [T]
The frequency of oscillation 'ω', with dimensions [1/T]
The height that the ball was dropped from 'H', with dimensions [L]
The phase of the plate 'Φ', with dimensions [T]
Gravity 'g', with dimensions [L/T²]
and the coefficient of restitution 'α', which was dimensionless

Because these variables and parameters all had dimensions in terms of length, time, both or neither, meaning that there were two dimensions in the system, two of the parameters could be removed via non-dimensionalization. This was due to the Buckingham-π theorem in dimensional analysis.

After a bit of deliberation, letting: \( T = \frac{2\pi}{\omega} \) be the period of oscillation, with dimensions [T], the following new dimensionless variables/parameters were introduced or kept:

\[
\tau = \frac{t}{T}, \quad X^* = \frac{2x}{gT^2}, \quad H^* = \frac{H}{A}, \quad \phi = \frac{\Phi}{T}, \quad \omega^* = \omega \sqrt{\frac{A}{g}}, \quad \alpha = \alpha
\]

This left g and A as the only two variables whose dimensions could not to be eliminated via non-dimensionalization, meaning that they were removed from consideration in the non-dimensionalized system. Also of interest in this problem was the velocity at which the ball was to leave the plate after each collision, \( v = \frac{dx}{dt} \). Using the non-dimensionalised variables \( X^* \) and \( \tau \) for position and time respectively, the non-dimensionalized velocity became: \( V^* = \frac{2v}{gT} \). Non-dimensionalizing velocity in this way meant that \( V^* \) basically corresponded to the number of periods of plate oscillation between collisions. This achieved neat results, as will be seen later. Ultimately, the non-dimensionalized velocity and non-dimensionalized phase were the two response variables that were most closely followed, with the non-dimensionalization of the 8 initial parameters/variables certainly making for neater results.

4 Computational Model

The computer program MatLab was utilized to model this system, with a variety of different codes that did different things developed over the six weeks. Ball and plate motion and ball-plate collision events were initially modelled and plotted using discrete timesteps, where collisions were determined by checking at which timestep the ball first fell below the plate and then using this as the collision time. However, this was a fairly crude approach to the situation, and the Newton-Raphson Method for root finding was adopted instead to more precisely and accurately determine numerical solutions. For equations like (3) in §3.1 and (6) in §3.2, the Newton Raphson method involves making an initial guess for t and then updating t with each iteration until f(t) is less than some small tolerance value. In this way, because equations such as those aformentioned cannot be analytically solved easily and an exact value is not generally required, the Newton Raphson method is good for providing numerical solutions to \( f(t) = 0 \) type equations with accuracy up to as many decimal places as the tolerance value.

Many attempts were made at developing a scheme for establishing an initial guess from which one could be sure that the Newton-Raphson method would zero in on the right root of the equation.
In all instances, this desired root was the first non-negative root. It was not desired, for example, for the Newton Raphson Method to zero in on a negative valued root in equation (3), as this would imply the next collision of the ball occurred backwards in time, which is obviously not realistic. Ultimately, after encountering a number of difficulties with more complex initial-guess schemes, it was decided to always make \( t_{\text{initial guess}} = t_n \) and then ensure that the solution \( t \) found satisfied \( t \geq t_{\text{initial guess}} \). In essence, this meant starting at the time of the previous ball-plate collision and stepping forwards in time until the next collision had occurred.

With this as the basis for the working computational model, a large variety of tests and results were able to be undertaken. Often these brought about further tweaks and amendments to the code, but ultimately the framework remained the same.

5 Experimental Approach

5.1 General Use of Model

Initially the model was used to plot ball and plate trajectories in time for any combination of values of the 6 non-dimensionalized variables/parameters. In the figure below, the plate’s vertical motion in time is shown in red while the ball’s motion is shown in blue. Green indicates that the ball has stuck to the plate and the two are moving together.

![Figure 2: Ball-Plate Trajectories in time for \( \alpha = 1, 0.5 \) and 0 respectively](image)

As can clearly be seen in Figure 2 above, for \( \alpha = 1 \) the ball maintains its bounce as little energy is lost, whereas for \( \alpha = 0.5 \) the ball quickly loses energy and for \( \alpha = 0 \) the ball just immediately sticks to the plate. The little bits of red in between times when the ball is stuck to the plate in green for the \( \alpha = 0 \) case indicate that the ball very briefly escapes from the plate before getting stuck once again. This always occurs just as the plate’s acceleration drops below gravity, which happens periodically; The ball escapes from the plate just before the plate reaches its maximum amplitude, sticks to the plate again almost immediately but lifts off once again one period later, and again for every period after that. As expected, for inelastic situations the ball could be expected to fall into some form of periodic motion almost immediately (in the case shown it happens on the first bounce, but in other instances this periodic motion may not be observed until after a number of bounces). Such periodic motion for other values of \( \alpha \) could be considered the chief point of interest in this project.
5.2 Focusing on the $\alpha = 0.99$ Case

In the early weeks of the project different values of $\alpha$ had been looked at and a variety of data had been collected. The coefficient of restitution was treated as a variable parameter during this time, but to reduce the parameter space further down from the 6 it was at after non-dimensionalization the decision was made to simply look at the one $\alpha$ case.

Now, there was already a considerable amount of existing literature concerning the inelastic case of this problem and partially inelastic cases with small coefficients of restitution (see for example Luck [3]). As a result, it was decided that the focus of investigation in the project would instead be on the long-term settling down of the ball to periodic motion for a high-$\alpha$, nearly elastic scenario. Rather than focusing on $\alpha = 1$, for which there is some conjecture concerning whether all possible circumstances of ball bouncing on plate end up in some kind of periodic motion, $\alpha$ was set to 0.99. It was thought that this would result in the system having attributes similar to that of the perfectly elastic case, but the small loss of energy associated with each collision would be enough to generate the appearance of interesting periodic motions.

Settling on this value for $\alpha$, different attempts were made at analysing the long term behaviour of the system. Code was generated that gave interesting plots such as the following.

![Figure 3: Non-dimensionalized time duration of bounces for $\omega = \pi$, $\alpha = 0.99$](image)

The above figure describes the durations of ball bounces for a particular frequency when the
ball is dropped from \( H^* = 30 \). As can be clearly seen, the system neatly moves towards being \( 2\)-periodic, where by the 700th bounce the ball is going back and forth between bouncing for just longer than 4 periods of plate oscillation and just less than 4 periods of oscillation. That is, in the long term every odd bounce occurs for the exact same length of time and every even bounce occurs for the exact same length of time, making the motion 2-periodic as the system stabilises.

While the kind of plot in Figure 3 was partly insightful into the way the computational system was behaving, it didn’t give a full picture as to what was going on. The above plot only indicated how long the ball was bouncing in the air for, not giving any insight into where the ‘stable’ bounces were occurring with respect to the plate’s motion. To get better insight into how the system was moving to periodicity, it was decided that the motion of the ball would be tracked in the \((\phi, V^*)\) phase plane, which would house all possible states that the system could be in for a given bounce.

### 5.3 Phase Plane Analysis

Mapping the ball’s motion in this phase space meant recording for each bounce \( n \) the phase \( \phi \) of the plate’s motion that the ball hit at and the speed \( V^* \) that the ball had upon leaving the plate. \( \phi \) ranges between 0 and 1, where 0 corresponds to 

\[
x_{\text{plate}} = A \sin(0 \times 2 \times \pi)
\]

0.25 corresponds to

\[
x_{\text{plate}} = A \sin(0.25 \times 2 \times \pi)
\]

etc. This of course means that \( \phi = 0 \) and \( \phi = 1 \) correspond to the same thing, meaning that the x-direction of the phase plane loops around.

A relationship between the non-dimensional values of the \( n\)th collision with that of the \((n + 1)\)th collision was what was desired in order to map the ball’s motion over time. This relationship could be found by first taking equation (2) from §3.1 and then taking the derivative to get the speed of the ball on impact with the plate \( u_{\text{ball}} \):

\[
u_{n+1} = \alpha(g(t_{n+1} - t_n) - v_n) + (1 + \alpha)u_{\text{plate}}
\]

Here \( t \) and \( v \) are replaced with \( t_{n+1} \) and \( v_{n+1} \) because solutions of this equation for \( t \) and \( v \) give the desired values of the \((n + 1)\)th collision. Taking the derivative of (1) to get \( u_{\text{plate}} \), then:

\[
u_{n+1} = \alpha(g(t_{n+1} - t_n) - v_n) + (1 + \alpha)A\omega \cos(\omega t_{n+1})
\]

Equations (7) and (8) are a coupled system of equations that describe the recursive relationship between the time and velocity of the ball for consecutive bounces, where basically:

\[
v_{n+1} = F(t_n, t_{n+1}, v_n) \quad (7) \quad \text{and} \quad t_{n+1} = G(t_n, t_{n+1}, v_n) \quad (8)
\]

The relationship can be re-expressed very easily in terms of \((\phi, V^*)\) by substituting in for \( t \) and \( v \) with their non-dimensional expressions from §3.3.
Thus, having found the recursive relationship linking one bounce to the next, phase plane images like the following were produced:

![Phase Plane Image](image)

Figure 4: Phase Plane for $\omega^* = 0.7$ starting at i.c (0.62, 4.22)

In the above, the ball’s motion is tracked for a number of bounces after it leaves the plate at some initial point with some initial velocity (coordinates $(\phi_0, V_0^*) = (0.62, 4.22)$). The initial height that it would have needed to be dropped from in order for this intial position to occur is not important in terms of the ball’s motion in the phase plane, which is a good thing as it effectively removes from consideration another parameter ($H^*$) from the parameter space. ($H^*$ is effectively wrapped up in $(\phi_0, V_0^*)$)

As can be seen from Figure 4, starting from the particular position in phase space shown has the ball moving rather randomly around for the first few bounces recorded. In a situation such as this, the system most likely will ultimately settle down to a periodic regime of some sorts, but it may take 100s of bounces, and will inevitably produce a phase plane full of dots with lines going in all directions. Thus, to get a handle on where in the phase plane periodic motion would be occurring, analytic solutions of this system were looked to.

5.4 Analytically Finding 1-Periodic Bouncing Schemes

Taking equations (7) and (8), one will have the ball bouncing in 1-periodic fashion if:

$$(\phi_n, V_n^*) = (\phi_{n+1}, V_{n+1}^*).$$

That is, the system will be 1-periodic when the ball is bouncing in exactly the same way every bounce. This can be described by letting:

$$(t_n, v_n) = (t_n + C \times T, v_{n+1}) = (t_{FP}, v_{FP})$$
where \( C \) is some positive integer ensuring that the ball hits at the same phase every time by making the duration of the bounce some whole number of periods. Substituting into equation (8) in §5.3 and rearranging, then:
\[

v_{FP} = \frac{gCT}{2} \tag{9}
\]

Using that \( V^* = \frac{2v}{gT} \) and hence \( v = \frac{gTV^*}{2} \), 1-periodic bouncing schemes only occur for \( V^* = C \). That is, because of the way \( V^* \) had been defined earlier, periodic orbits would only occur for some \( \phi \) at integer values of \( V^* \). Substituting this result back into equation (7), then:
\[

\left( \frac{1 - \alpha}{1 + \alpha} \right)V_{FP}^* = A\omega \cos(2\pi\phi_{FP})
\]

and hence;
\[

\phi_{FP} = \frac{1}{2\pi} \arccos\left( \frac{(1 - \alpha)V_{FP}^*}{(1 + \alpha)A\omega} \right) \tag{10}
\]

Thus, for a given frequency and at the values of \( \alpha \) and that has already being specified, using (10) one could find where all the 1-periodic regimes in phase space would be, with \( V_{FP}^* = C \), where \( C \in \mathbb{N} \).

6 Phase Space Exploration

6.1 A 1-Periodic regime in Phase Space
For the randomly chosen frequency 0.7, and starting near where a 1-periodic bouncing scheme was predicted at \((\phi, V^*) = (0.2347, 3)\) (given by (10)), the following phase-space plot was produced:

![Phase Plane for \( \omega^* = 0.7 \) starting at i.c (0.4,3)](image)

Figure 5: Phase Plane for \( \omega^* = 0.7 \) starting at i.c (0.4,3)
As can be seen in Figure 5, starting in the phase plane near where periodicity is expected results in a neat spiral pattern in towards that particular point. In the above only the first 100 bounces have been recorded and shown (for image clarity), but when the simulation is run for a longer number of bounces the system zeros right in on the analytically expected \((0.2347,3)\). This spiralling inwards (known as a 'foci' in dynamical systems phase-space terminology) indicates that the 1-periodic point is stable. That is, when the system is in a state close enough to continually bouncing at \((0.2347,3)\), which has a ball-plate trajectory plot looking like:

![Figure 6: 1-Periodic motion corresponding to continual bouncing at \((0.2347, 3)\), with 3 oscillations of the plate occurring between every bounce as \(V^* = 3\), and collisions occurring at \(A \sin(0.2347 \times 2 \times \pi)\)](image)

Anything in phase space within that particular critical point’s region of attraction will after some finite number of bounces end up at the critical point bouncing in the same way as above. The region of attraction for \((0.2347,3)\) just includes all points in phase space that, if started at, will have the system end up going to the critical point in some number of finite bounces.

### 6.2 Other Types of Periodic Bouncing

1-periodic bouncing wasn’t the only type of scheme that the ball-plate system could tend towards. After much experimenting in the phase plane, the ball could also be found to settle into regimes such as the following:
Depending where the system was initialised from, (ie: \((\phi_0, V_0^*)\) for the first bounce), after some number of bounces the system could end up stabilising to any of the types of schemes shown in Figure 7. Recorded below are all the corresponding critical points found in the phase plane below \(V^* = 5\) for \(\omega^* = 0.7\):

Figure 8: All the critical points found for 0.7 below \(V^* = 5\)
Here each circle corresponds to a stable critical point; a point in phase space corresponding to some kind of periodic motion, to which certain other nearby points tend towards. The colours of each match up to trajectories shown earlier: critical points corresponding to 1-periodic motion are black, 2-periodic are dark blue, 3-periodic are magenta, 4-periodic are grey, 5-periodic are light green and 8-periodic are light blue. The dotted lines have been included to partition off groups of periodic orbits, and also to help indicate how the structure is repeated moving up the phase plane. For example, the 5-periodic motion shown in Figure 7 corresponds to moving in a repeated fashion between the 5 green circles in the 2nd lowest section of the plane (near $V^* = 1$). That is, at the $n^{th}$ collision the ball will leave the plate with $(\phi, V^*)$ of one of the green circles. At the next bounce, it will be at another of the green circles, then another after that, and at the $(n + 5)^{th}$ collision (5 bounces later) it will be back at the same $(\phi, V^*)$ as it was for $n$.

The same 5-periodic scheme is repeated about all 5 integer values of $V^*$, as are the 1-periodic and 2-periodic cycles, whereas some of the higher number periodic schemes were only found below $V^* = 1$. It is important to remember though that this picture may not be complete, as this image was only generated by starting at a large selection of random initial positions in the phase plane and observing what critical points were tended towards. Some exotic types of periodicity may not have been found at this particular frequency, as their corresponding critical points in phase space may have had very small regions of attraction that none of the initial guesses made ended up falling into.

It is also important to note that all the circles shown represent foci similar to that shown in Figure 5. If one were to only plot $(\phi, V^*)$ for every 5th bounce for the 5-periodic motion, then one would get a single spiral in towards a critical $(\phi, V^*)$ value. Basically, all $n$ types of bounces in a stable $n$-periodic motion are also stable, and nearby types of bounces will zero in on this motion.

Finally, the crosses shown in Figure 8 correspond to the complements of the analytically found 1-periodic stable points in §5.4. These points result from the fact that when one solves equation (10) in §5.4, the presence of the arccos term results not only in the existence of the stable solution at $\phi_{FP}$ but a similarly periodic but unstable solution at $1 - \phi_{FP}$. The nature of these and similar points will be discussed in section §7.2.

### 6.3 Going Backwards in Time

Of interest were the regions of attraction for these periodic orbits; for a given stable periodic regime (set of the same coloured critical points that the ball bounces between), from what regions in the phase plane would the ball ultimately end up tending to this motion? To determine this, a model for going backwards in time was developed. This meant finding a relationship between the $n^{th}$ bounce and the $(n - 1)^{th}$ bounce. Performing a little bit of algebra on equations (7) and (8) in §5.3, then;

\[
v_{n-1} = \frac{1}{\alpha}(g(t_{n-1} - t_n) - v_n) + (1 + \frac{1}{\alpha})A\omega \cos(\omega t_{n-1})
\]

\[
A\sin(\omega t_{n-1}) = -\frac{g}{2}(t_{n-1} - t_n)^2 + v_n(t_{n-1} - t_n) + A\sin(\omega t_n)
\]
This coupled systems of equations links the $n^{th}$ bounce and the $(n-1)^{th}$ bounce together, allowing for one to trace back where the ball would previously have had to bounce in order for it to end up at bounce $n$. Using this model, and starting from a number of 'final' guesses near the critical points of the 8-periodic motion about $V^* = 0.5$ in Figure 8, the following plot was produced.

Figure 9: Region of attraction for the 8-Periodic orbit about $V^* = 0.5$

Adding in regions of attraction for two other periodic regimes but decreasing the number of points plotted for the sake of clarity:

Figure 10: Regions of attraction for 1-Periodic (blue), 5-Periodic (green) and 8-Periodic (red) schemes
These two plots give an interesting picture of what was going on in the system. As can be seen for the plot of the region of attraction for the 8-period scheme in Figure 9, containing upwards of 10000 points, there are many places in which the ball could bounce that result in it ending up in the 8-periodic scheme. All the black dots in the plot are \((\phi, V^*)\) collisions that have been traced back to after starting at or very very near the 8-periodic regime around \(V^* = 0.5\); that is, starting at any of the black dots in the image will have the system move to the 8-periodic scheme after a finite number of bounces.

Of note is the fact that there are clear bands of white. These indicate the presence of other periodic schemes with their own regions of attraction (schemes such as those shown in Figure 8); these regions remain white because all \((\phi, V^*)\) points in this area tend to those critical points and not the 8-periodic scheme.

There are other regions of white as well as the obvious white bands (such as those at \(\phi = 0.75\) for half integer values of \(V^*\), which correspond to the regions of attraction for the dark-blue 2-cycles in Figure 8). Due to the time constraints imposed by only conducting this research over a 6-week period, not all such regions have been investigated. This an area for further investigation to go along with the search for more periodic schemes.

Figure 10 is useful for indicating what is happening between roughly \(\phi = 0.5\) and \(\phi = 0.9\), as well as in between the white bands. It is clear that there exists a region of mixing; if a collision occurs where the ball bounces with a \((\phi, V^*)\) in this colourful and at first glance chaotic region, it can basically be said that the ball is still in the process of looking for a periodic regime to 'fall into' or tend towards. It is believed that with enough resolution this region may have fine structure and order to it, so although it certainly appears to be chaotic, again further investigation needs to be conducted to determine the region's nature (see [3]).

Rather than focusing further inquiry into these regions at this rather random frequency of plate oscillation \(\omega^* = 0.7\), it was felt that there was perhaps more to gleaned from examining how parts of the general picture changed as \(\omega^*\) was varied.

## 7 Bifurcations Due to Changing the Frequency \(\omega^*\)

### 7.1 Stability Analysis for 1-Periodic Behaviour

Because an effectively infinite space was being dealt with (‘infinite’ not only because \(V^*\) ranged between 0 and \(\infty\) but because there could exist many critical points with miniscule regions of attraction that had not yet been found), there was really only enough time to examine the behaviour of one of the types of periodicity so far observed. Because analytic 1-periodic solutions had already been found and confirmed to match with the computational model, the 1-periodic solution was the one that was looked at. As was seen in Figure 5 in §6.1, the critical point corresponding to the 1-periodic orbit analytically determined by equation (10) was a foci (spiral). Investigation would center on how the nature of this spiral changed as \(\omega^*\) was increased.

In order to analytically determine the nature of the critical point (ie: why it was an inwards spiral for \(\omega^* = 0.7\)), a stability analysis needed to be performed. This involved taking the recursive
relationship described by (7) and (8) in §5.3 and making the following perturbations to \(\phi\) and \(V^*\) terms:

\[
(\phi_n = \phi_{FP} + \epsilon \hat{\phi}_n, \quad V^*_n = V^*_{FP} + \epsilon \hat{V}^*_n) \quad \text{and} \quad (\phi_{n+1} = \phi_{FP} + \epsilon \hat{\phi}_{n+1}, \quad V^*_{n+1} = V^*_{FP} + \epsilon \hat{V}^*_{n+1})
\]

Here the values for the \(n^{th}\) and \((n+1)^{th}\) bounce are set to the same thing (in line with the system being at a 1-periodic fixed point), but this time with some small perturbation factor introduced for each term, with \(\epsilon\) to fall out of the calculation when the stability analysis is performed. Substituting the perturbed terms above into (7), linearising the equation by removing any terms of \(O(\epsilon^2)\) or higher (since these terms will be small and are ignored in a linear stability analysis) and then cancelling the terms on either side that fall out because they satisfy equation (10):

\[
\epsilon \hat{V}^*_{n+1} + \epsilon \alpha \hat{V}^*_n = 2\epsilon (\alpha - (\omega^*)^2(1 + \alpha) \sin(2\pi \phi_{FP})) \hat{\phi}_{n+1} - 2\epsilon \alpha \hat{\phi}_n \quad (13)
\]

Likewise, substituting the perturbed terms into (8) and cancelling the terms that make (9):

\[
\epsilon V^*_{FP} \hat{V}^* = \epsilon (\omega^* \cos(2\pi \phi_{FP}) + V^*_{FP})(\hat{\phi}_{n+1} - \hat{\phi}_n) \quad (14)
\]

Cancelling the epsilons and rearranging the results for (13) and (14) into a matrix, the relationship between the \(n^{th}\) and \((n+1)^{th}\) perturbations is described by:

\[
\begin{bmatrix}
2(\omega^*)^2(1 + \alpha) \sin(2\pi \phi_{FP}) - 2\alpha & 1 \\
\omega^* \cos(2\pi \phi_{FP}) + V^*_{FP} & 0
\end{bmatrix}
\begin{bmatrix}
\hat{\phi}_{n+1} \\
\hat{V}^*_{n+1}
\end{bmatrix} =
\begin{bmatrix}
-2\alpha & -\omega^* \cos(2\pi \phi_{FP}) - V^*_{FP} \\
\omega^* \cos(2\pi \phi_{FP}) + V^*_{FP} & V^*_{FP}
\end{bmatrix}
\begin{bmatrix}
\hat{\phi}_n \\
\hat{V}^*_n
\end{bmatrix}
\]

Pre-multiplying both sides of the matrix equation above by the inverse of the left hand side, the matrix system can be re-expressed as:

\[
\begin{bmatrix}
\hat{\phi}_{n+1} \\
\hat{V}^*_{n+1}
\end{bmatrix} = A
\begin{bmatrix}
\hat{\phi}_n \\
\hat{V}^*_n
\end{bmatrix}
\]

The above relationship describes how the initial perturbations made to the critical point behave as the system updates itself at each bounce. If the perturbations get smaller as the number of bounces \(n\) gets larger, then the critical point will be stable, but if the perturbations increase then the critical point will be unstable and will correspond to a periodic-bouncing scheme that the ball-plate system will not want to settle at.

The way the perturbations behave for a given 1-periodic critical point \((\phi_{FP}, V^*_{FP})\) for certain \(\omega^*\) is determined by the eigenvalues of the matrix \(A\). If the magnitudes of both eigenvalues are less than 1 then the critical point will be stable (ie: the \((n+1)^{th}\) perturbations will be smaller than the \(n^{th}\) perturbations, meaning that the perturbations will die away to zero). If one of the magnitudes is greater than 1 however, the perturbations will decrease and the critical point will be unstable.

### 7.2 Bifurcations for the \(V^* = 3\) 1-Periodic Foci

Applying the above stability analysis to the 1-periodic foci previously found at \((0.2347,3)\) for \(\omega^* = 0.7\), it was found that \(A\) had eigenvalues 0.0174 + 0.9898i and 0.0174 - 0.9898i. The magnitude of each of these eigenvalues is 0.99, which is less than 1 and is hence consistent with the critical point and 1-periodic behavior being stable. The presence of imaginary parts in the eigenvalues is
what results in the critical point being a foci; that is, the spiralling behaviour seen in the phase plane near (0.2347,3) is a result of the eigenvalues being complex. In short, the computationally derived picture of the phase plane near the critical point at (0.2347,3) was consistent with the analytic linear stability analysis used above.

To get a glimpse at how the system began to behave differently for different $\omega^*$, the same stability analysis was performed at the critical point at $V^* = 3$ for a range of $\omega^*$ values. The following data was collected:

![Bifurcation diagram for varied $\omega^*$](image)

- Stable anti-clockwise foci
- Stable clockwise foci
- Unstable Saddle Point

Figure 11: Behaviour of critical point at $V^* = 3$ as $\omega^*$ is varied (with small period-doubling regions enlarged below)
Before going into the plot above, it should be noted that this analysis was done for 1-periodic critical points from $V^* = 1$ right up to $V^* = 10$, with the same behaviour as shown for 3 exhibited at each of these integer values, but with the picture shifted slightly to the right for each integer $V^*$ going up.

From Figure 11 it can be seen that a number of bifurcations occurred for the system as $\omega^*$ was varied, where a bifurcation is a drastic change in behaviour brought about by a small change in parameter value. Below $\omega^* = 0.217$, it was found that no critical point existed for $V^* = 3$. That is, there was no solution to equation (10) for $V^* = 3$. In terms of the physics of the ball-plate system, this meant that if the frequency of plate oscillation was too low there was not enough energy present for the ball to get into a regime where it bounced exactly every 3 periods of plate oscillation. When performing this analysis for $V^* = 1$, it was found that no critical point existed below $\omega^* = 0.126$. This meant that this value of 0.126 was the critical value of plate oscillation frequency in the system, below which no stable 1-periodic bouncing schemes would occur.

Above $\omega^* = 0.217$ a critical point did exist for $V^* = 3$, with a $\phi$ coordinate that started at 0 (ie:when the plate was moving upwards with maximum velocity) before asymptoting as $\omega^*$ increased towards $\phi = 0.25$ (where the velocity of the plate is zero and no energy is given to the system). At $\omega^* = 0.217$ the eigenvalues of $A$ were $0.99 \pm 0i$. As $\omega^*$ was increased, the real part of these eigenvalues decreased down through zero at $\omega^* = 0.707$ to -0.99 at $\omega^* = 0.997$, while the imaginary part was such that the magnitude of each eigenvalue was always 0.99 for any $\omega^*$ in this range. That is, at $\omega^* = 0.217$ and $\omega^* = 0.997$ the imaginary part was 0, while at $\omega^* = 0.707$ the imaginary part was $\pm 0.99$. It was not discovered why the value 0.99 kept coming up, but it was a very interesting result considering $\alpha$ was set to 0.99 earlier on.

The point at which the real part of the eigenvalues went through zero marked the first bifurcation of the system. Examining the foci around the particular frequency that this bifurcation occurred at by tracing out a series of approach curves from initial guesses near this point, the following set of images were produced:
As can be seen in Figure 11, as the frequency passed through the bifurcation value the direction of approach towards the critical point changed from counter-clockwise to clockwise. This was a result of the sign change in the real part of the eigenvalues of A at this value. Also of interest is the fact that when the foci reversed direction a 4-period regime appeared around it (i.e. The 4 spirals about the center spiral for 0.72 and 0.73). Again, it is not known why this 4-periodic orbit appeared just as the bifurcation occurred, but it is certainly a point of interest for further research.

Going back to Figure 11, the 1-periodic critical point was seen to remain a stable clockwise foci until $\omega^*$ reached 0.997, at which point the real part of each eigenvalue was -0.99 and the imaginary part was 0. Above this $\omega^*$ where the imaginary part of the eigenvalues for A went through zero, eigenvalues ceased to have imaginary part and instead were completely real. For any $\omega^*$ beyond this point, one eigenvalue was less than -1 and the other was between -1 and 0. Having one eigenvalue with magnitude less than 1 and the other greater than 1 indicated instability, and was in fact consistent with the presence of a saddle point. This is the same type of point as those complementary 1-periodic points marked with a cross in Figure 8 in §6.2. They basically look like a $y = \frac{1}{x}$ hyperbola plots in phase space, where there is attraction towards the critical point along one axis but then repulsion away from the point along the other, making the point unstable.

It was discovered that bifurcations involving a stable point turning into an unstable one are often pitchfork bifurcations. However, this type of bifurcation involves the creation of two new stable points at the time of the bifurcation, points which move away from the now unstable one. Looking in the same region of the phase plane where the $V^* = 3$ 1-periodic point existed but now at $\omega^* = 1$, a pair of 2 such stable critical points was discovered (a period-2 regime between which the ball was bouncing). As the frequency was increased further, these points were traced, and as seen in Figure 11, another pitchfork bifurcation occurred at $\omega^* = 1.100$, which then resulted in a 4-periodic regime. Tracing these points further (seen in the enlarged close-ups of the plot), two further pitchfork bifurcations were observed, resulting in 8-periodic and then 16-periodic behaviour. This doubling from 1-periodic to 2-periodic to 4-periodic and so on is a common phenomenon known as period doubling. At each bifurcation point, the doubling of the number of critical points corresponds to a
doubling in the number of different ball bounce’s the system will be going through before it repeats its motion, meaning that the period of the regime doubles (that is, the number of plate oscillations between repeating the bounce pattern doubles).

Due to precision issues, it became very difficult to track the system beyond \( \omega^* = 1.119 \) as it moved to 32-periodic and so on. This was because as each pitchfork bifurcation occurred, the region of attraction for each particular critical point became much smaller. In essence, in order to fall into the 16-periodic regime, one basically had to start exactly in the regime to begin with, otherwise the ball-plate system would move off to some other scheme.

Despite this, using the data collected a comparison was able to be made with the Feigenbaum constant, which is a transcendental value associated with period-doubling in a vast array of dynamical systems. The Feigenbaum constant gives the ratio between the duration (in terms of the parameter that is being varied (\( \omega^* \))) of \( 2^n \)-periodicity and \( 2^{n+1} \)-periodicity for \( n \geq 1 \). It should tend to 4.6692 as \( n \to \infty \). Using the values at which bifurcations occurred at in Figure 11, it was found that:

\[
\frac{T_2}{T_4} = \frac{1.100 - 0.997}{1.1182 - 1.100} = 5.66
\]

That is, the ratio between the range of \( \omega^* \) values for which 2-periodicity was observed compared with 4-periodicty was 5.56. Because the system could only be traced up until it became 16-periodic, further terms in this sequence (ie: \( \frac{T_4}{T_8}, \frac{T_8}{T_{16}}, \ldots \)) could not be generated. However, with more time and further adjustments to the way data was collected, it could be expected that this series of ratios would converge to 4.6692 given that the first term found was not particularly far from it.

8 Conclusion

Ultimately, a large variety of interesting phenomena and results were gleaned from the MatLab model created for a ball bouncing up and down on an oscillating plate. Computational values matched those that were ascertained analytically, the presence of stable 1-periodic regimes matched expectations for a nearly perfectly elastic system, and it was pleasing to find that the system exhibited period-doubling behaviour as the frequency was increased. However, it was surprising to both observe the presence of higher order periodic regimes and to see how complex the phase space picture was for any given frequency of plate oscillation. With more time, there are a number of further actions that could be taken with regards to this project, including:

- Finding higher \( \frac{T_{2^n}}{T_{2^{n+1}}} \) ratios to see how well the period-doubling scheme converges to the Feigenbaum constant
- Investigating what happens to the 4-periodic scheme brought about by the bifurcation at \( \omega^* = 0.707 \)
- Understanding the relationship between the eigenvalues of A and the value of \( \alpha \) (both had magnitude 0.99 for stable regimes)
- Extending the method of stability analysis for 1-periodic behaviour to n-periodic behaviour
• Exploring more of the phase plane; determining where the more exotic critical points go and come from as $\omega$ is varied, and getting a more complete picture of the various regions of attraction

• Determining how things change for other values of $\alpha$ aside from $\alpha = 0.99$

• Applying the ideas used for this 1-dimensional system to the more complex scenario of a ball bouncing on an oscillating infinite see-saw (a couple of weeks were spent looking at this system, but the model remains incomplete)

• Extending this model to match the physics of bouncing fluid droplets

9 Acknowledgements

I would like to thank the Australian Mathematical Sciences Institute (AMSI) and Monash University for providing scholarship funding and the opportunity for me to undertake this project. I thoroughly enjoyed the 6 weeks of vacation research, and recommend the program highly to anyone else considering undertaking it. I would also like to give a massive thankyou to my supervisor Anja, for not only generously dedicating a significant portion of her time to my project, but for providing me with fantastic insight into the world of academia and research.

References

