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The Ricci curvature of rotationally symmetric metrics

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1 Introduction

Riemannian geometry is a branch of differential non-Euclidean geometry developed by Bernhard Riemann, used to describe curved space. In Riemannian geometry, a *manifold* is a topological space that locally resembles Euclidean space. This means that at any point on the manifold, there exists a neighbourhood around that point that appears 'flat'and could be mapped into the Euclidean plane. For example, circles are one-dimensional manifolds but a figure eight is not as it cannot be projected into the Euclidean plane at the intersection. Surfaces such as the sphere and the torus are examples of two-dimensional manifolds.

The shape of a manifold is defined by the *Riemannian metric*, which is a measure of the length of tangent vectors and curves in the manifold. It can be thought of as locally a matrix valued function. The *Ricci curvature* is one of the most significant geometric characteristics of a Riemannian metric. It provides a measure of the curvature of the manifold in much the same way the second derivative of a single valued function provides a measure of the curvature of a graph. Determining the Ricci curvature of a metric is difficult, as it is computed from a lengthy expression involving the derivatives of components of the metric up to order two. In fact, without additional simplifications, the formula for the Ricci curvature given by this definition is essentially unmanageable.

$\mathbb{R}^n$ is one of the simplest examples of a manifold. This report will focus on Riemannian metrics on $\mathbb{R}^n$ that are $SO(n)$-invariant (unchanged under rotations of $\mathbb{R}^n$) as they have the property that they are conformal to a flat metric. This means that, in $\mathbb{R}^n$, the angles between vectors with respect to these metrics are the same as the angles between vectors with respect to a metric with zero curvature. A consequence of this is that the formula for the Ricci curvature on this class of metrics simplifies significantly and several important differential equations for Riemannian metrics (such as the prescribed Ricci curvature equation, the Ricci flow and the Einstein equation) become tractable. In particular, the fundamental problem of recovering a metric from its Ricci curvature (the prescribed Ricci curvature problem) reduces to a single ordinary differential equation, as shown by Cao & DeTurck (1994) in their paper 'The Ricci curvature equation with rotational symmetry.'

This report will first provide an introduction to some of the basic concepts in Riemannian geometry. It will then outline how the arguments of Cao & DeTurck reduce the prescribed Ricci curvature problem for rotationally symmetric metrics on $\mathbb{R}^n$ to a single ordinary differential equation and explain how that is applied to find a sufficient condition for the global existence of a solution.

2 Riemannian Geometry

The definitions stated in this section are from the book, *Riemannian Geometry* (Carmo, 1993).

Differentiable Manifold

A manifold is a topological space that is locally Euclidean, or locally flat. From an intuitive point of view, it can be thought of as a natural extension of a surface to arbitrary dimensions. A surface is an example of a two-dimensional manifold which we typically imagine embedded in three-dimensional space - a manifold is an n-dimensional space that is treated as independent of the ambient space surrounding it.

Formally, a *manifold* M of dimension n is a topological space which satisfies the following conditions:

(i) M is a Hausdorff space.

(ii) M has a countable basis.

(iii) For every point $p \in M$ there exists an open neighbourhood U that is homeomorphic to an open subset Ω of $\mathbb{R}^n$.

Such a homeomorphism $\mathbf{x}\colon U \to \Omega$ is called a *(coordinate) chart.* An *atlas* is a family of charts, $\{U_\alpha, \mathbf{x}_\alpha\}$ for which the U_α constitute an open covering of M. Local coordinates provide a systematic method for locally representing a manifold in such a way that calculations can be carried out. For a chart $(U, \mathbf{x})$ the local coordinates of the manifold M will be expressed as $\mathbf{x} = (x_1, ..., x_n)$.

A *differentiable manifold* is one with a differentiable structure defined at every point, allowing us to apply many of the techniques from multivariable calculus to the manifold. Formally, a 'differentiable structure' means that for a maximal atlas on the manifold M, for any pair α, β, with $U_\alpha \cap U_\beta \neq \varnothing$, the sets $\mathbf{x}_\alpha(U_\alpha \cap U_\beta)$ and $\mathbf{x}_\beta(U_\alpha \cap U_\beta)$ are open sets in $\mathbb{R}^n$ and the mappings $\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}\colon \mathbf{x}_\alpha(U_\alpha \cap U_\beta) \to \mathbf{x}_\beta(U_\alpha \cap U_\beta)$ are differentiable of class C^∞. The mappings $\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}$ are known as *transition maps*, and because they map $\mathbb{R}^n \to \mathbb{R}^n$, differentiation can be defined in the usual way on Euclidean space.

Tangent Vector

Next we would like to extend the notion of a tangent vector and the tangent space to differentiable manifolds. The idea of a tangent space is pretty clear for a surface that is embedded in $\mathbb{R}^3$ - say we have a sphere, we can picture a tangent space at a point as simply a plane that sits on the sphere perpendicular to the radius. A tangent vector is then some element in the tangent space. But if we want to generalise this notion to an arbitary curved space of n-dimension, then we need a new intrinsic definition that depends only on the manifold and not the ambient space it is embedded in.

Let M be a differentiable manifold. A differentiable function α: $(-\epsilon, \epsilon) \to M$ is called a *(differentiable) curve* in M. A *tangent vector* at $p \in M$ is an equivalence class of differentiable curves α with $\alpha(0) = p$. Clearly, the curves are on the manifold and not in the ambient space, which fulfils the requirement for an intrinsic definition. Two curves α_1 and α_2 are equivalent (i.e. $\alpha_1 \equiv \alpha_2$) iff

$$\left.\frac{d(f \circ \alpha_1)}{dt}\right|_{t=0} = \left.\frac{d(f \circ \alpha_2)}{dt}\right|_{t=0} \quad \text{for any } f \in \mathcal{D}$$

where $\mathcal{D}$ is the set of all functions of class C^∞ at p on M. The collection of all tangent vectors at p forms a vector space, known as the *tangent space* of M at p, denoted T_pM. So if a tangent vector is an equivalence class of curves on the manifold, the tangent space at a point is just the collection of all equivalence classes of curves through that point.

An alternative interpretation of a tangent vector is as an operator, $\alpha'(0) : \mathcal{D} \to \mathbb{R}$, given by

$$\alpha'(0)f = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}$$

If we choose a chart $(U, \mathbf{x})$ at $p \in U$, we can express the function f and the curve α in local coordinates by

$$f \circ \alpha(t) = f \circ \mathbf{x}^{-1} \circ \mathbf{x} \circ \alpha(t) = f(x_1(t), ..., x_n(t))$$

Thus, restricting f to α, we obtain

$$\begin{aligned} \alpha'(0)f &= \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0} = \left.\frac{d}{dt} f(x_1(t), ..., x_n(t))\right|_{t=0} \\ &= \sum_{i=1}^{n} x_i'(0) \left(\frac{\partial f}{\partial x_i}\right) = \left(\sum_i x_i'(0) \left(\frac{\partial}{\partial x_i}\right)\right) f \end{aligned}$$

Therefore, the tangent vector $\alpha'(0)$ can be represented as the operator

$$\alpha'(0) = \left(\sum_i x_i'(0) \left(\frac{\partial}{\partial x_i}\right)\right)$$

in local coordinates. The choice of chart $(U,\mathbf{x})$ determines an *associated basis* $\left\{\left(\frac{\partial}{\partial x_i}\right),...,\left(\frac{\partial}{\partial x_n}\right)\right\}$ in the tangent space T_pM.

Vector Fields

A *vector field* X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, X is a mapping of M into the tangent bundle TM, where $TM = \{(p,v)|p \in M, v \in T_pM\}$. The vector field is *differentiable* if the mapping $X : M \to TM$ is differentiable.

Considering a chart $(U,\mathbf{x})$ we can write

$$X(p) = \sum_{i=1}^{n} a_i(p)\frac{\partial}{\partial x_i}$$

where each $a_i : U \to \mathbb{R}$ is a function on U and $\left\{\frac{\partial}{\partial x_i}\right\}_{i=1,...,n}$ is the basis of T_pM associated to $(U,\mathbf{x})$. Thus, X is differentiable iff the functions a_i are differentiable for some chart.

It is also useful to think of a vector field as a mapping $X : \mathcal{D} \to \mathcal{D}$ defined by

$$(Xf)(p) = \sum_{i=1}^{n} a_i(p)\frac{\partial f}{\partial x_i}(p)$$

As such, X applied to f at p gives a measure of the rate in which f changes at p in the direction $X(p)$ (i.e. it can be thought of as analogous to the directional derivative of a function in calculus).

Interpreting X as an operator on $\mathcal{D}$ allows us to consider the application of a vector field to another vector field. For example, if X and Y are differentiable fields on M and f is a differentiable function on M we can consider the functions $X(Yf)$ and $Y(Xf)$. In general, these iterative operations will not lead to vector fields as they involve derivatives of order higher than one. However, in the case $Z(f) = X(Yf) - Y(Xf)$ the second order terms cancel and $Z = [X,Y]$ is a unique vector field called the *bracket* of X and Y.

Tensors

Let us indicate by $\mathcal{X}(M)$ the set of all vector fields of class C^∞ on M and by $\mathcal{D}(M)$ the ring of real-valued functions of class C^∞ on M. A *k-tensor* T on a Riemannian manifold M is a multilinear mapping

$$T : \underbrace{\mathcal{X}(M) \times \ ... \ \times \mathcal{X}(M)}_{\text{k factors}} \to \mathcal{D}(M)$$

This means that, given $Y_1, ..., Y_k \in \mathcal{X}(M)$, $T(Y_1, ..., Y_k)$ is a differentiable function on M and T is linear in each argument:

$$T(Y_1, ..., fX + gY, ..., Y_k) = fT(Y_1, ..., X, ..., Y_k) + gT(Y_1, ..., Y, ..., Y_k)$$

for all $X, Y \in \mathcal{X}(M), f, g \in \mathcal{D}(M)$. The number k is called the *order* of T.

Denote the set of all k-tensors on M by $T^k(M)$ (Note $T^k(M)$ is a vector space). Let M be a Riemannian manifold and let $S \in T^k(M)$ and $T \in T^l M$. We can define a map:

$$S \otimes T : \underbrace{\mathcal{X}(M) \times \ ... \ \times \mathcal{X}(M)}_{\text{k+l factors}} \to \mathcal{D}(M)$$

by

$$S \otimes T(X_1, ..., X_{k+l}) = S(X_1, ..., X_k)T(X_{k+1}, ..., X_{k+l})$$

$S \otimes T$ depends linearly on each argument X_i separately so it is a $(k+l)$-tensor called the *tensor product* of S and T.

Let $S \in T^2(M)$ be a 2-tensor on a manifold M. We can represent S in the chart $(U, \mathbf{x})$ by

$$S = \sum_{i,j} S_{ij} dx^i \otimes dx^j$$

where S_{ij} are components of the tensor S and dx^i is the differential covector field for the local coordinate x^i. Assume S is symmetric ($S_{ij} = S_{ji}$). The symmetric tensor S applied to two tangent vectors $v, w \in T_p M$ with coordinate representations $v = (v_1, ..., v_n)$ and $w = (w_1, ..., w_n)$, (i.e. $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$ and $w = \sum_{j=1}^n w_j \frac{\partial}{\partial x_j}$) is

$$\begin{aligned} S(u, v) &= \sum_{i,j} S_{ij}(dx^i \otimes dx^j)(u, v) \\ &= \sum_{i,j} S_{ij} dx^i(u) dx^j(v) \\ &= \sum_{i,j} u^i v^j S_{ij} \end{aligned}$$

Riemannian Metric

It is useful to introduce metric structures on differentiable manifolds, as it makes it possible to define a variety of geometric notions on a manifold. These may include: angles between tangent vectors, lengths of curves, curvature, gradients of functions and divergence of vector fields. In a vector space the metric structure is usually given by a scalar product. On a differentiable manifold we thus define a

Riemannian metric as follows:

A *Riemannian metric* on a differentiable manifold M is a tensor that associates to each point p of M an inner product $\langle\ ,\ \rangle$ (that is, a symmetric, bilinear, positive definite form) on the tangent space T_pM, which varies differentiably from point to point.

So if we have two vectors on the tangent space at a point, the Riemannian metric maps those two vectors onto $\mathbb{R}$ by way of an inner product.

We can equip any differentiable manifold with a Riemannian metric and it becomes a *Riemannian manifold*, and that is enough to describe essentially any geometric notions about the manifold that we want to know.

In the chart $(U, \mathbf{x})$ at $p \in M$ a metric is represented by a positive definite, symmetric matrix

$$(g_{ij}(\mathbf{x}(p)))_{i,j=1,\ldots,n}$$

such that $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$.

Thus, the standard Euclidean metric on $\mathbb{R}^2$ can be represented by the matrix $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or in the tensor notation outlined above by $g = dx \otimes dx + dy \otimes dy = dx^2 + dy^2$.

Levi-Civita Connection

Say we want to differentiate a vector field at some point on a manifold in a particular direction. Then we need to compare two vectors induced by the field: one at the point at which we want to find the derivative; and one close-by in the direction we are differentiating in. However, the two vectors at these nearby points will lie in different tangent spaces and we cannot, in general, compare them. Thus, if we want to take the directional derivative of a vector field, we require some mathematical structure on the manifold that 'connects' tangent spaces and allows us to compare vectors.

An *affine connection* ∇ on a differentiable manifold M can be thought of as a way of differentiating a vector field with respect to another vector field. It is a mapping

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

denoted by $(X, Y) \to \nabla_X Y$ and which satisfies the following properties:

(i) $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$.

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$, in which $X, Y, X \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$.

An affine connection ∇ on a differentiable manifold M can be used to define a differential operator on the manifold. We can show that there exists a unique correspondence which associates to a vector field V along the differentiable curve $c: I \subset \mathbb{R} \to M$ another vector field $\frac{DV}{dt}$ along c, called the *covariate derivative* of V along c, such that:

(a) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$.

(b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, where W is a vector field along c and f is a differentiable function on I.

(c) If V is induced by a vector field $Y \in \mathcal{X}(M)$, i.e. $V(t) = Y(c(t))$, then $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$.

Once equipped with an affine connection, we have the means to compare tangent vectors on the manifold at different points. This is achieved via parallel transport.

Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c: I \to M$ is called *parallel* when $\frac{DV}{dt} = 0$, for all $t \in I$. If $c: I \to M$ is a differentiable curve in M and V_0 is a vector tangent to M at $c(t_0)$, $t_0 \in I$ (i.e. $V_0 \in T_{c(t_0)}M$), then there exists a unique parallel vector field V along c, such that $V(t_0) = V_0$. $V(t)$ is called the *parallel transport* of $V(t_0)$ along c. It can be thought of as the 'velocity' at t of the curve c with initial velocity V_0 at t_0.

An affine connection ∇ on a smooth manifold M is said to be *symmetric* when

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \text{for all} \quad X, Y \in \mathcal{X}(M).$$

An affine connection is said to be *compatible* with the Riemannian metric of M when for any vector fields V and W along the differentiable curve $c: I \to M$ we have

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle, \quad t \in I$$

The choice of a Riemannian metric on a manifold M uniquely determines a certain affine connection ∇ on M known as the *Levi-Civita* (or *Riemannian*) *connection*, satisfying the following conditions:

(a) ∇ is symmetric

(b) ∇ is compatible with the Riemannian metric

Geodesic

A geodesic is a curve that minimises the distance (as defined by the Riemannian metric) between two nearby points on a Riemannian manifold. If M is a Riemannian manifold, with Riemannian connection ∇ then a curve $\gamma : I \to M$ is a *geodesic at* $t_0 \in I$ if $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ at the point t_0; if γ is a geodesic at t, for all $t \in I$, we say that γ is a *geodesic*. This is equivalent to saying that a geodesic is a curve that is capable of parallel transporting its tangent vector.

Riemann Curvature and Sectional Curvature

The *Riemann curvature* R of a Riemannian manifold M is a correspondence that associates to every pair $X, Y \in \mathcal{X}(M)$ a mapping $R(X,Y) : \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad Z \in \mathcal{X}(M)$$

where ∇ is the Riemannian connection of M. It can be thought of as measuring the non-commutativity of ∇. If $M = \mathbb{R}^n$ then $R(X,Y)Z = 0$ for all $X, Y, Z \in \mathcal{X}(\mathbb{R}^n)$, so R also indicates how much the manifold M deviates from Euclidean space. Note that R is bilinear in $\mathcal{X}(M) \times \mathcal{X}(M)$ and the curvature operator $R(X,Y) : \mathcal{X}(M) \to \mathcal{X}(M)$ is linear.

Given a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_pM$ with basis $\{x, y\}$, the real number

$$K(x,y) = K(\sigma) = \frac{\langle R(x,y)x, y\rangle}{|x \wedge y|}$$

is called the *sectional curvature* of σ at p. Note $|x \wedge y| = \sqrt{\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2}$.

Ricci Curvature and Scalar Curvature

Let $x = z_n$ be a unit vector in T_pM; we can take an orthonormal basis $\{z_1, ..., z_{n-1}\}$ of the hyperplane in T_pM orthogonal to x and consider the following averages:

$$Ric_p(x) = \frac{1}{n-1} \sum_i \langle R(x, z_i)x, z_i\rangle, \quad i = 1, 2, ...n-1$$

$$K(p) = \frac{1}{n} \sum_j Ric_p(z_j), \quad j = 1, ..., n.$$

These expressions are called the *Ricci curvature* in the direction x and the *scalar curvature* of p, respectively. The Ricci curvature is the average of the sectional curvatures of all planes in T_pM containing x, and the scalar curvature is the average of the Ricci curvatures of all unit vectors (i.e. it is the average of the sectional curvatures of all planes in T_pM).

The bilinear form given by

$$\begin{aligned}\mathrm{Ric}(x,y) &= \frac{1}{n-1}\ trace(z \to R(x,z)y) \\ &= \frac{1}{n-1}\sum_{i}^{n-1}\langle R(x,z_i)y, z_i\rangle,\end{aligned}$$

where $x, y \in T_pM$, is known as the *Ricci tensor* (i.e. $\mathrm{Ric} : \mathcal{X}(M)\times\mathcal{X}(M) \to \mathcal{D}(M)$). Like the Riemannian metric, the Ricci curvature is a matrix-valued function on the tangent space at every point in the manifold.

Another, more intuitive description of Ricci curvature is as follows: Let v be a unit tangent vector at some point $p \in M$. Let C be any small neighbourhood of M having an arbitrary shape. For each point $z \in C$, consider a geodesic $t \to z_t$ starting at z with initial velocity v (using parallel transport to identify v with a tangent vector at z). Moving along the geodesics, they will not remain parallel but rather, will get closer together or further apart depending on the curvature of the manifold. If we 'slide' C along these geodesics (i.e. let C_t be the set $\{z_t \mid z \in C\}$), then C will deform, affecting its volume. The amount by which the volume deviates from its starting volume is dependent upon the Ricci curvature of the manifold in the direction v:

$$\mathrm{vol}\ C_t = \mathrm{vol}\ C(1 - \frac{t^2}{2}\mathrm{Ric}(v) + \text{smaller terms})$$

Thus we can say that Ricci curvature in a particular direction on a manifold controls the evolution of volumes under geodesic flow in that direction. Clearly, if we were performing the same procedure in Euclidean space - sliding a shape along parallel straight lines - this would have no effect on the volume of the shape, so the Ricci curvature of a flat metric is zero.

3 Equation of Prescribed Ricci Curvature

Overview

The prescribed Ricci curvature problem can be summarised as follows: Given a symmetric 2−tensor T defined on a manifold of dimension $n \geq 3$, can a metric g be found so that T is the Ricci curvature tensor of g? The definition of Ricci curvature transforms this question into the problem of finding a solution to a nonlinear system of second-order partial differential equations. We denote this system of equations by

$$\mathrm{Ric}(g) = T$$

Whether or not the equation of prescribed Ricci curvature has a solution has been a significant question in geometric analysis since the early 1980's. The ultimate goal is to determine global results on existence, uniqueness and regularity of metrics with prescribed Ricci tensors, however this has yet to be achieved. DeTurck took the first major step towards achieving this goal in his paper [5], where he determined when one can solve the equation $\mathrm{Ric}(g) = T$ locally, in a neighbourhood of a point $p \in M$. Formally, he obtained the following result:

If T is a C^∞ tensor in a neighbourhood of a point $p \in M$ and $T^{-1}(p)$ exists (i.e. T is nondegenerate at p), then there exists a C^∞ Riemannian metric g such that $\mathrm{Ric}(g) = T$ *in a neighbourhood of p.*

Although a local solution to the problem is possible, we may not be able to solve $\mathrm{Ric}(g) = T$ globally on all M. A strong nonexistence theorem was produced by DeTurck and Koiso in their paper [6] which stated that, whenever T is positive-definite, there is a constant $c_T > 0$ such that $c_T T$ is not the Ricci curvature of any Riemannian metric on M.

Whether a solution to $\mathrm{Ric}(g) = T$ is unique in any sense has been tackled in many papers and some progress has been made, however as of today, a complete answer is still lacking.

Clearly, solving this problem is not a simple matter and in fact, most of the work that has been done on the Prescribed Ricci curvature problem focuses on exploring different settings and simplifications that can be made to the system in order to make the problem tractable. One example of a setting that does make the problem tractable was proposed by Cao and DeTurk in [2]. In this case, the authors looked at the simplest example of a manifold, $\mathbb{R}^n$, and restricted the metrics they chose to a special class known as rotationally invariant metrics. With these two simplifications, the authors found that we can reduce the Prescribed Ricci curvature system of second-order partial differential equations to a single ordinary differential equation, and then they used this to develop a sufficient condition for the global existence of a solution, as well as to prove the uniqueness of the solution.

The Ricci curvature equation with rotational symmetry on $\mathbb{R}^n$

We will now discuss Cao & DeTurck's paper and the key result obtained.

Note, when working on $\mathbb{R}^n$, we let (t, Θ) be polar coordinates on $\mathbb{R}^n$, so that Θ represents coordinates on the unit $(n-1)$-sphere S^{n-1} and we let $d\Theta^2$ be the canonical metric on S^{n-1}.

A symmetric covariant 2-tensor on $\mathbb{R}^n$ will be called *rotationally invariant* if it remains invariant under the action of $SO(n)$, where $SO(n)$ is the *special orthogonal group:*

$$SO(n) = \{A \in M_{n\times n} \mid A^T = A^{-1};\ \det(A) = 1\}$$

Thus T is rotationally invariant if $T = \sigma T \sigma^{-1}$ for all $\sigma \in SO(n)$. Intuitively, we can think of rotational symmetry of a tensor in the following way: Say we have a tensor T on a manifold M at a point $p \in M$. If we rotate the manifold M by some angle Θ, then the rotationally symmetric tensor T at p will remain unchanged after rotation. We can represent our rotationally symmetric tensor in polar coordinates by

$$T = \alpha(t) dt \otimes dt + \beta(t) d\Theta \otimes d\Theta$$

where the functions α and β are real-valued and depend only on the radial position t of the point p.

The primary result of Cao and DeTurck's paper [2] is the following theorem:

If the smooth, nonsingular, rotationally symmetric tensor $T = \varphi(t)dt^2 + t^2\phi(t)d\Theta^2$ *satisfies*

$$\frac{d(t^2|\phi(t)|)}{dt} > 0$$

for all $t > 0$*, then* $\mathrm{Ric}(g) = T$ *has a rotationally symmetric solution defined on all of* $\mathbb{R}^n$

This result is achieved by using the conformal properties of rotationally invariant tensors, along with a transformation that simplifies the Ricci system, in order to show that a global solution exists. The authors first show that the rotationally invariant metric on $\mathbb{R}^n$ is conformally flat, which allows them to derive a formula for the Ricci tensor of the metric. They then introduce a transformation called *the Ricci potential* to simplify the Ricci system to a single first order nonlinear ordinary differential equation (ODE) - which when solvable, implies that the Ricci system is solvable. Local existence and uniqueness of solutions to the Ricci potential ODE are then studied, before finally showing a sufficient condition for the existence of a global solution to the ODE and thus, a sufficient condition for the existence of a global solution to the Ricci system.

We will now review the findings from the paper in more detail, highlighting the key points and briefly outlining some of the proofs. The first lemma we address is as follows:

Lemma: *Suppose* $T = \varphi(t)dt^2 + t^2\phi(t)d\Theta^2$ *is a smooth, nondegenerate rotationally symmetric 2-tensor on* $\mathbb{R}^n$*. Then* T *is either positive or negative definite everywhere.*

If we take a unit vector v at the origin with respect to the standard Euclidean metric g_0 then it is easy to see $T(v,v) = \varphi(0)$. So T is positive or negative definite at the origin dependent on the sign of $\varphi(0)$. As T is nondegenerate everywhere (which means the eigenvalues of $T \neq 0$ for all t) and φ and ϕ are real, continuous functions (and they are the eigenvalues of T), they will not change sign. So T is positive definite or negative definite everywhere.

The fact that a smooth, nondegenerate rotationally symmetric 2-tensor is either positive or negative definite everywhere is then used to show that any smooth, rotationally symmetric metric g on $\mathbb{R}^n$ can be expressed as

$$g = e^{2f(t)}[(r'(t))^2dt^2 + r^2(t)d\Theta^2] \tag{1}$$

where $r(t)$ satisfies $r(0) = 0$, $r'(0) = 1$ and $r'(t) > 0$ for all t ≥ 0.

From (1) we can see that g is conformal to the flat metric. Conformality of metrics is an equivalence relation - two Riemannian metrics g and g' on a manifold M are said to be conformal if there exists a C^∞ function f on M such that $g = e^{2f}g'$. We can think of conformal changes to a metric as angle-preserving transformations. Clearly, (1) above is in the form $g = e^{2f}g_0$ where $g_0 = dr^2 + r^2d\Theta^2$ is the standard Euclidean metric (note: $dr = r'(t)dt$ so $dr^2 = (r'(t))^2dt^2$).

Determining that g is conformal to a flat metric means we can derive the Ricci tensor for g using the standard formula for the Ricci tensor after a conformal change of metric (obtained from [1], p58):

$$\mathrm{Ric}(g) = \mathrm{Ric}(g_0) - (n-2)(\nabla df - df \circ df) + (\triangle f - (n-2)\|df\|^2)g_0$$

where ∇df is the Hessian with respect to g_0 and $\triangle f$ is the Laplacian with respect to g_0. From this we get the following lemma:
Lemma: *The Ricci tensor of the metric*

$$g = e^{2f(r)}[dr^2 + r^2d\Theta^2]$$

is

$$Ric(g) = \alpha(r)dr^2 + r^2\beta(r)d\Theta^2,$$

where $\alpha(r) = -(n-1)[f_{rr} + f_r/r]$ *and* $\beta(r) = -[f_{rr} + (2n-3)f_r/r + (n-2)(f_r)^2]$. *Note:* $f_r = f'(t)/r'(t)$ *and* $f_{rr} = f_r'(t)/r'(t)$.

At this point we can see that we are trying to solve the Ricci system $\mathrm{Ric}(g) = T$ in the rotationally symmetric context, where we are given $T = \varphi(t)dt^2 + t^2\phi(t)d\Theta^2$ and we seek g in the form (1). We do not know the function $r(t)$ in advance (however, we do know $f(t) = log(h_2(t)) - log(r(t)/t)$ where $h_2^2(t) = \phi(t)$ - this was derived in the paper). So the Ricci system in the rotationally symmetric case is two second-order nonlinear ODEs with one unknown:

$$\alpha(r)(r')^2 = \varphi \tag{2}$$

$$\beta(r)r^2 = t^2\phi \tag{3}$$

The Ricci potential w_g is a function introduced by the authors which is used to transform the equations (2) and (3) into a single first-order ODE. It is a function of the Gauss curvature of a fixed two-dimensional linear subspace of $\mathbb{R}^n$ with respect to the metric g. Substituting w_g into (2) and (3) results in the ODE

$$\left(\frac{dw}{dt}\right)^2 = \frac{1}{n-1}[(n-2)\varphi(t)(w^2 - 2w) + t^2\varphi(t)\phi(t)] \tag{4}$$

with initial conditions $w(0) = w'(0) = 0$.

Equation (4) greatly simplifies the analysis of the Ricci system $\mathrm{Ric}(g) = T$. The authors determine that if the rotationally symmetric 2-tensor $T = \varphi(t)dt^2 + t^2 d\phi(t)d\Theta^2$ is nondegenerate everywhere and if (4) has a solution, then the Ricci system $\mathrm{Ric}(g) = T$ is solvable.

Using the equivalence of the Ricci potential ODE (4) with the Ricci system $\mathrm{Ric}(g) = T$, the authors go on to prove the local existence of a solution to the system. Uniqueness of a solution is proven using the standard uniqueness theorem for ODEs. Finally, a global solution to the ODE problem is found to exist with the sufficient condition $\frac{d(t^2|\varphi(t)|)}{dt} > 0$ - leading to the main result of the paper.

References

[1] Besse, A. (1987). *Einstein manifolds.* Berlin: Springer-Verlag.

[2] Cao, J. and DeTurk, D. (1994). The Ricci Curvature Equation with Rotational Symmetry. *American Journal of Mathematics*, 116(2), p.219.

[3] Carmo, M. (1993). *Riemannian geometry.* Boston: Birkhauser.

[4] DeTurck, D. (1980). The equation of prescribed Ricci curvature. *Bull. Amer. Math. Soc.*, 3(1), pp.701-705.

[5] DeTurck, D. (1981). Existence of metrics with prescribed Ricci curvature: Local theory. *Inventiones Mathematicae*, 65(2), pp.179-207.

[6] DeTurck, D. and Koiso N. (1984). Uniqueness and nonexistence of metrics with prescribed Ricci curvature. *Annales de l'institut Henri Poincar (C) Analyse non linaire* , 1, pp.351-359.

[7] Jost, J. (2008). *Riemannian geometry and geometric analysis.* Berlin: Springer.