Cycle Spectrum for Honeycomb Toroidal Graphs

Joshua Connor

Supervised by Brian Alspach and Thomas Kalinowski

The University of Newcastle

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1 Introduction

The existence of cycles in graphs has been a major area of study since the very beginning of graph theory. The properties of graphs we examine in this paper are pancyclic and girth. A graph G is *pancyclic* if it contains cycles of length $r \in \{3, 4, \dots V\}$ where V is the order of G. The girth of a graph G is the length of the shortest cycle in G.

Although a graph is not strictly pancyclic unless it has girth 3, it is still useful to conisder the spectrum of the cycles it does contain. As you will see, the class of graphs we are about to examine are bipartite, and thus they contain no odd cycles. Hence a bipartite graph G can be *even pancyclic* if it contains all even length cycles in the set $\{4, 6, \dots 2[V/2]\}$.

In 1996, Bogdanowicz [1] was able to show that a connected circulant graph of degree at least three contain all even cycles, and that if a connected circulant graph has girth three then it contains cycles of all lengths. In a related paper, Alspach, Bendit and Maitland [2] were able to show that a connected Cayley graph of degree at least 3 on an abelian group is *even edge-pancyclic.* Even edge-pancyclic occurs when every edge is in even cycles of every possible length. In another paper by Alspach and Muir [3] it was shown that on a generalised dihedral group, a connected, bipartite graph X may be even edge-pancyclic. The following class of graphs arose as a road block in extending these theorems.

The class of graphs in question are the *honeycomb toroidal graphs.* These graphs are parameterised by the number of columns m, rows n, and the size of the *jump* edge they contain ℓ. As such they are denoted HTG(m, n, ℓ). Honeycomb toroidal graphs are undirected, contain no loops and no multiple edges, and are 3-regular.

Definition 1. Beginning with the definition of the honeycomb toridal graph, the vertex set is defined as follows. There are mn vertices arranged as an $n \times m$ array, where m is a positive integer and $n \geq 4$ is even. We coordinatise the vertices as follows:

$$\{(i, j) : 0 \leq i \leq m - 1 \text{ and } 0 \leq j \leq n - 1\}.$$

We refer to the vertices with a fixed first coordinate as a *column* and the vertices with a fixed second coordinate as a *row.* We order the first coordinates of the vertices left to right, and the second coordinates bottom to top.

Definition 2. The edges are more complex and are defined as follows. We include the edges joining (i, j) to $(i, j+1)$ for all $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, where we treat the second subscript modulo n. In other words, each column is an n-cycle. We will refer to these as vertical edges. The edges between the columns are a bit more complicated. When i is even and $i < m - 1$, then there is an edge from (i, j) to $(i + 1, j)$ for all even j. When i is odd and $i < m-1$, then there is an edge from (i, j) to $(i+1, j)$ for all odd j, which are refered to as horizontal edges. Finally, all that is left is the *jump* edge ℓ. This edge reconnects the column $m-1$ to the column 0. When the number of columns m is even, then ℓ is also even, and it takes the vertices $(m - 1, j)$ to $(0, j + \ell)$ for all odd j. When the number of columns m is odd, then we have ℓ odd, and it takes the vertices $(m - 1, j)$ to $(0, j + \ell)$ for all even j (for all even j in the $m - 1$ column). An example of a honeycomb toroidal graph is illustrated in Figure 1.

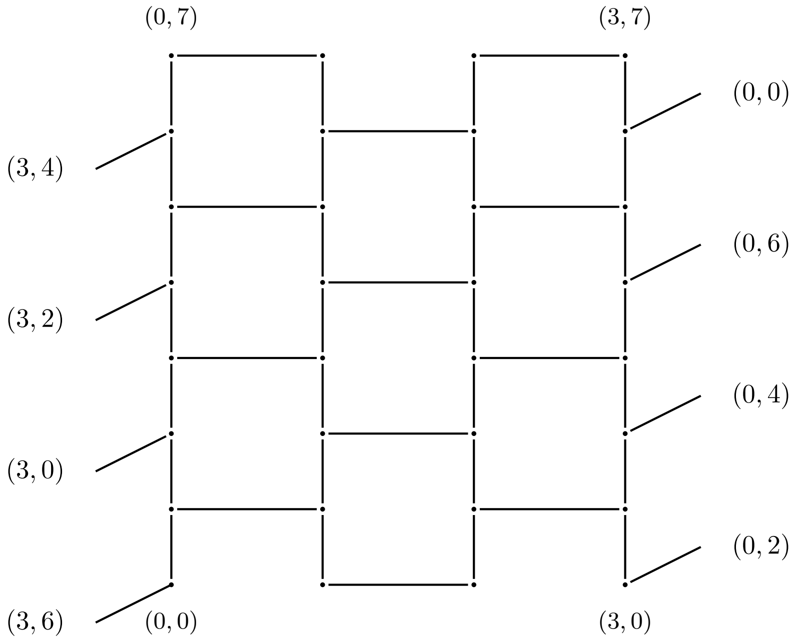

Figure 1: The honeycomb toroidal graph HTG(4, 8, 2).

2 Even Pancyclic Results

Now that we have a definition for the graph, let us prove an important property that will simplify our research.

Proposition 1. *Honeycomb toroidal graphs are bipartite.*

Proof. Separate the vertics into two subsets: $u(i,j)$ where $i+j$ is even, and $v(i,j)$ where $i+j$ is odd. It is clear that no horizontal or vertical edge connects any two vertices of the same subset. It is also true that no vertical edge from $(i, n-1)$ to $(i, 0)$ violates this either, as each column begins wither either an even vertex or an odd vertex, and since each column is of even height it must finish with the a vertex from the other subset.

To see that the jump also maintains this, observe that idependent of whether we have an odd number of columns or an even number, the vertices in the last column, that are incident with a jump edge are always in the subset $u(i,j)$, while those that are incident with a jump edge, in the first column, are in $v(i,j)$.

To see that those vertices incident to a jump edge in the column $m-1$ are always in the subset $u(i,j)$, where $i+j$ is even, observe the following. If m is even then $m-1$ is odd, and only vertices in the set $(m-1, j)$ where j is odd are incident to a jump edge (from the definition). Therefore $m-1+j$ is even. If m is odd then $m-1$ is even, and likewise j is even, thus $m-1+j$ is even. Also from the definition we have that only the vertices $(0, j)$ will be incident to a jump edge for all odd j. Hence we see that the jump edge only ever connects even vertices (from the last column) to odd vertices (in the first column). An example can be seen in Figure 2. □

Now that we know honeycomb toroidal graphs are bipartite, we can reduce our workload to exploring only even length cycles.

We will split our cycles into two catagories: cycles of length $r \equiv 2 \pmod 4$ and those of length $r \equiv 0 \pmod 4$. First we introduce some definitions that will aid us in proving our results.

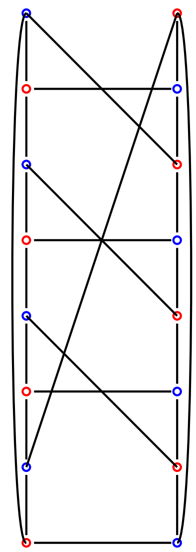

Figure 2: The graph HTG$(2, 8, 2)$ illustrating that it is 2-colourable.

Definition 3. Let C be a cycle in HTG(m, n, ℓ) containing the horizontal edge $e = \{(i, j),\ (i + 1, j)\}$. If $V(C) \cap \{(i, j + 1), (i, j + 2), (i + 1, j + 1), (i + 1, j + 2)\} = \emptyset$, then we obtain a new cycle of length $|V(C)| + 4$ by replacing the edge e with the 5-path

$$(i, j),\ (i, j + 1),\ (i, j + 2),\ (i + 1, j + 2), (i + 1, j + 1),\ (i + 1, j).$$

This is called the *vertical upward bypass* on the edge $e = \{(i, j),\ (i + 1, j)\}$. A variation to this process is called the *vertical downward bypass*, which is defined similarly but where the second coordinates will be $j - 1$ and $j - 2$.

Definition 4. Let C be a cycle in HTG(m, n, ℓ) containing the vertical edge $e = \{(i, 1),\ (i, 2)\}$, where the column i is the right most column of C, and C contains none of the edges of the 5-path $(i, 2),\ (i + 1, 2),\ (i + 1, 1),\ (i + 1, 0),\ (i, 0),\ (i, 1)$. We can replace e by the above 5-path to obtain a new cycle of length $|V(C)| + 4$. This is called the *horizontal bypass* on the edge $e = \{(i, 1),\ (i, 2)\}$.

Definition 5. Let C be any cycle of length $r \equiv 2 \pmod 4$ and let P be the 5-path $\{(i, j{+}1), (i, j{+}2), (i{+}1, j{+}2), (i{+}1, j{+}1), (i{+}1, j), (i, j)\}$ both in HTG(m, n, ℓ). If $C \cap \{(i, j), (i, j{+}1)\} = \emptyset$, then replace P with the edge $\{(i, j), (i, j + 1)\}$ to create a new cycle of length $|V(C)| - 4$. This is called a *reverse horizontal bypass*.

Using these methods we are able to prove the following theorem.

Theorem 1. *Honeycomb toroidal graphs with an even number m of columns contain all cycles of length $r \equiv 2 \pmod 4$, where $6 < r < mn$.*

Proof. Since m is even and n is even, we have $mn \equiv 0 \pmod 4$ and hence the largest possible cycle on HTG(m, n, ℓ) will be $mn - 2$, such that it has length $r \equiv 2 \pmod 4$,

To show that these particular honeycomb toroidal graphs contain cycles of all lengths $r \equiv 2 \pmod 4$ consider the following construction. We begin with HTG(m, n, ℓ) and draw in the intial 6-cycle $C = \{(0, 1), (0, 2), (0, 3), (1, 3), (1, 2), (1, 1)\}$.

We now wish to collect the remaining cycle lengths. To do this we perform an *upwards vertical bypass* on the edge {(0,3),(1,3)} to obtain the next cycle of length $|V(C)| + 4$. Iterate this method repeatedly on the top horizontal edge, until all desired cycles contained by HTG$(2, n, \ell)$, for particular n, are obtained.

To extend $m = 2$ to $m = 4$ take the edges $\{(0,1),(1,1)\}$ and $\{(0,n-1),(1,n-1)\}$ and subdivide each edge twice resulting in the two 3-paths $\{(0,1),(x,1),(y,1),(1,1)\}$ and $\{(0,n-1),(x,n-1),(y,n-1),(1,n-1)\}$. Insert an x-column and a y-column consisting of the vertices $(x,0),(x,1),\ldots,(x,n-1)$ and $(y,0),(y,1),\ldots,(y,n-1)$. Now replace the $\{(x,1),(y,1)\}$ with the 3-path $\{(x,1),(x,0),(y,0),(y,1)\}$ and the edge $\{(x,n-1),(y,n-1)\}$ with the 3-path $\{(x,n-1),(x,n-2),(y,n-2),(y,n-1)\}$.

The above procedure produces a cycle C of length $2n+6$. If we now re-coordinatize the vertices by changing x to 1, y to 2, and 2 in the first coordinate to 3, then the cycle C belongs to HTG$(4,n,\ell)$. We now use successive downward vertical bypasses starting with the edge $\{(1,n-2),(2,n-2)\}$ to obtain cycles of length $2n+6, 2n+10, \ldots, 4n-2$.

The longest cycle has length $4n-2$ when $m = 4$. Note that this longest cycle contains the two horizontal edges $\{(0,1),(1,1)\}$ and $\{(0,n-1),(1,n-1)\}$ so that we may repeat the same procedure for $m = 6$. It is easy to see that for each even m we have the same two horizontal edges in the longest cycle. Thus, for every even m, the preceding procedure gives cycles of all lengths $r \equiv 2 \pmod 4$ except $r = 2n+2, r = 4n+2, \ldots, r = (m-2)n+2$. This procedure is demonstrated in Figures 3-8.

We now demonstrate how to pick up these missing lengths. Begin with the edge $\{(m-2,n-1),(m-1,n-1)\}$ and perform a negative vertical bypass. That is, remove the 5-path $\{(m-2,n-3),(m-2,n-2),(m-2,n-1),(m-1,n-1),(m-1,n-2),(m-1,n-3)\}$ and insert the edge $\{(m-2,n-3),(m-1,n-3)\}$. This produces a cycle of length $|V(C)|-4$. Continue this process repeatedly until you are left with the 5-path $\{(m-2,2),(m-2,3),(m-1,3),(m-1,2),(m-1,1),(m-2,1)\}$ and perform a reverse horizontal bypass. Repeat this process of collapsing horizontal edges, and reversing horizontal bypasses, until you have collected all cycles of lengh $r \equiv 2 \pmod 4$ where $6 \le r \le mn$. This is equivalent to reversing the steps to find the largest length cycle, except on a larger number of columns. $\square$

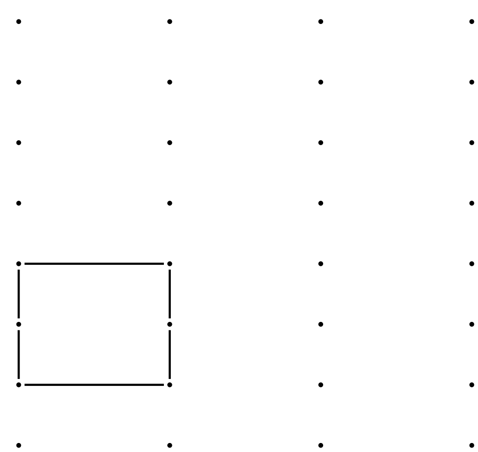

Figure 3: A cycle of length 6 in HTG$(4,8,\ell)$.

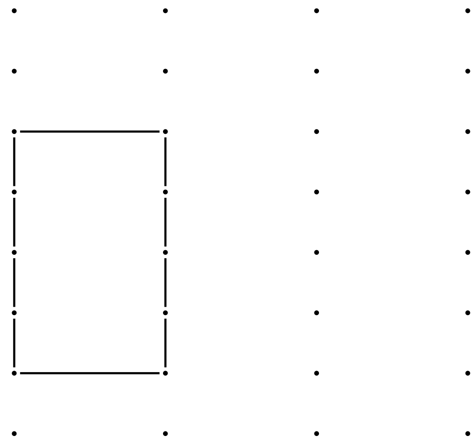

Figure 4: A cycle of length 10 in $\mathrm{HTG}(4, 8, \ell)$.

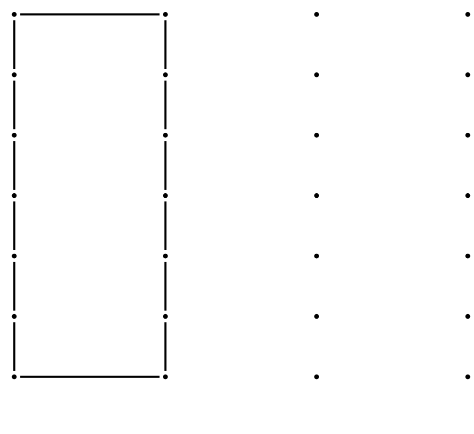

Figure 5: A cycle of length $2n-2$ in $\mathrm{HTG}(4, 8, \ell)$.

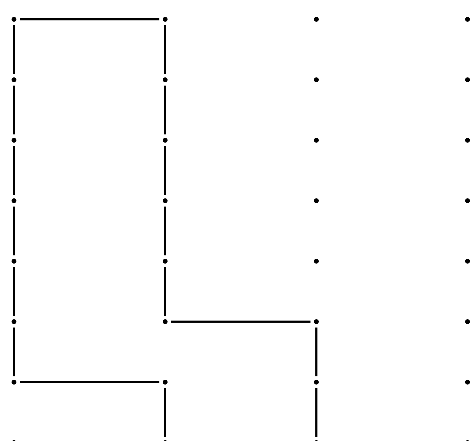

Figure 6: A horizontal bypass on the edge $\{(1,1),(1,2)\}$.

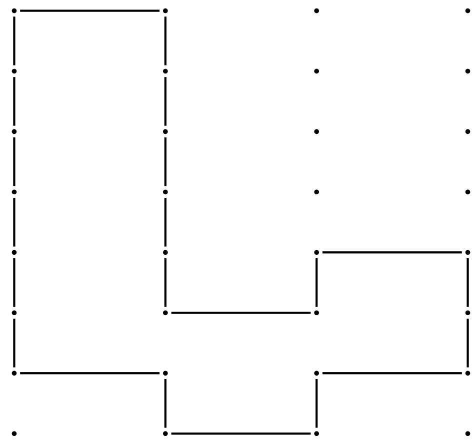

Figure 7: A horizontal bypass on the edge $\{(2,1),(2,2)\}$.

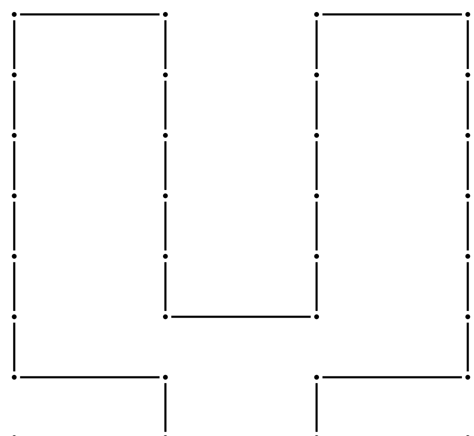

Figure 8: A cycle of length $mn-2$ in HTG$(4,8,\ell)$.

So now we know that if the number of columns m is even, then the honeycomb toroidal graph contains all cycles of length $r \equiv 2 \pmod 4$. What if the number of columns is odd?

Theorem 2. *Honeycomb toroidal graphs with an odd number of columns m, contain all cycles of length $r \equiv 2 \pmod 4$, such that $6 < r \leqslant mn$.*

It is actually true that when $m = 1$ they also contain these cycle lengths, but it is easier to prove separately. Again we introduce some definitions in order to show this.

Definition 6. Take the edge $\{(m-2,2),(m-2,1)\}$ and replace it with the path, $\{(m-2,2),(m-1,2),(m-1,3)\ldots(m-1,n-2),(m-1,n-1),(m-1,0),(m-2,0),(m-2,1)\}$. This is called the *terminating path* when $mn \equiv 0 \pmod 4$. This is illustrated in Figure 9.

Definition 7. Take the 2-path $\{(m-2,2),(m-2,3),(m-2,4)\}$ and replace it with the path, $\{(m-2,4),(m-1,4),(m-1,5)\ldots(m-1,n-1),(m-1,0),(m-1,1),(m-1,2),(m-2,2)\}$. This is called the *terminating path* when $mn \equiv 2 \pmod 4$. This is illustrated in Figure 10.

Note that the proof is very similar to the m even case and will draw some ideas from it.

Proof. In order to prove that all honeycomb toroidal graphs, with an odd number of columns, contain all cycles of length $r \equiv 2 \pmod 4$ we must split this into two situations: when $mn \equiv 0 \pmod 4$ and when $mn \equiv 2 \pmod 4$. Beginning with the base case HTG$(3, n, \ell)$, start with the 6-cycle $\{(0,1),(0,2),(0,3),(1,3),(1,2),(1,1),(0,1)\}$ and vertically bypass the edge $\{(0,3),(1,3)\}$ upwards repeatedely until all cycles of length $r \equiv 2 \pmod 4$ are collected within the first two columns. This gives us all cycles of length $r \equiv 2 \pmod 4$ from $r = 6$ to $r = 2n - 2$.

To extend the graph horizontally we replace the edge $\{(1,2),(1,1)\}$ with the appropriate terminating path. To collect all remaining cycles, vertically bypass the edge $\{(1,3),(2,3)\}$ upwards untill all cycles have been obtained. In completing this terminating path you may notice that some cycles of specific lengths are repeated. However, once you complete the downwards vertical bypasses, all cycles of length $r \equiv 2 \pmod 4$ will have been collected.

To extend m subdivide the edges $\{(0,1),(1,1)\}$, and $\{(0,n-1),(1,n-1)\}$ twice, inserting an $x - column$ and a $y - column$ at each of the points of subdivision. Redraw the edges and recoordinatise the vertices and described previously. Then vertically bypass the edge $\{(1,n-2),(2,n-2)\}$ until all extra cycles have been obtained. This allows us to take a graph, in which all cycles of length $r \equiv 2 \pmod 4$ are obtained, increase the number of columns by two and then proceed to obtain the new remaining cycles of length $r \equiv 2 \pmod 4$.

Note that when $mn \equiv 2 \pmod 4$, which only occurs when m is odd and $n \equiv 2 \pmod 4$, that we expect there to be a Hamilton cycle as this cycle is also congruent to $2 \pmod 4$. This is exactly the case and can be proven.

Employing the theorem that any edge in a 3-regular graphs lies in an even number of Hamilton cycles [4], we have that there must exist a Hamilton cycle that contains at least one jump edge. Once we have found such a cycle we can use this to prove that there exists a Hamilton cycle of length $r \equiv 2 \pmod 4$.

Take the base case $m = 1$ and find the Hamilton cycle that uses at least jump edge. Detach the jump edges from the vertex $(0, j + \ell)$ and insert the desired number of columns to the right of the original column. Dettach the jump edges from the vertics $(0, j)$ for all j odd, and reattach the jump edges to their corresponding vertices in the last column. To collect the remaining vertices in the graph insert an edge from $(0, j)$ to $(1, j)$ for all j that previously had a jump edge. Follow the vertical path from $(1, j)$ until you arrive at the vertex $(1, j')$, where $(2, j' + 1)$ has a jump edge. Move from $(1, j')$ to $(2, j')$ and then downwards until to reach $(2, j)$. Take the jump edge to $(0, j + \ell)$ and continue the process until the cycle is completed. This construction will collect a Hamilton cycle on HTG(m, n, ℓ) such that $mn \equiv 2 \pmod 4$. A Hamilton cycle where $mn \equiv 2 \pmod 4$ is shown in Figure 11. □

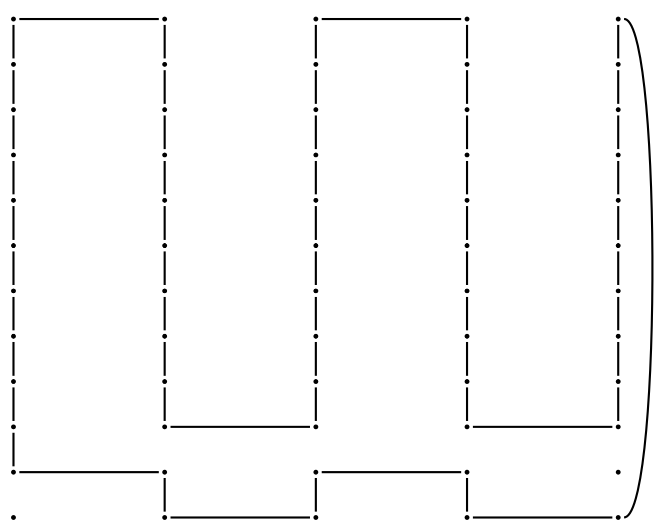

Figure 9: A terminating path in HTG$(5, 12.\ell)$, where $mn \equiv 0 \pmod 4$.

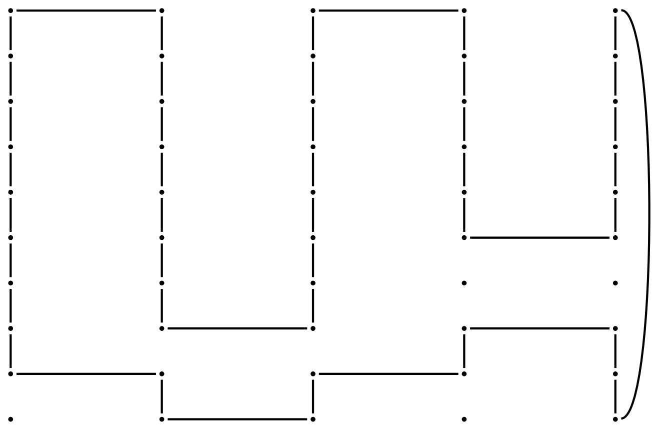

Figure 10: A terminating path in HTG$(5, 10, \ell)$, where $mn \equiv 2 \pmod 4$.

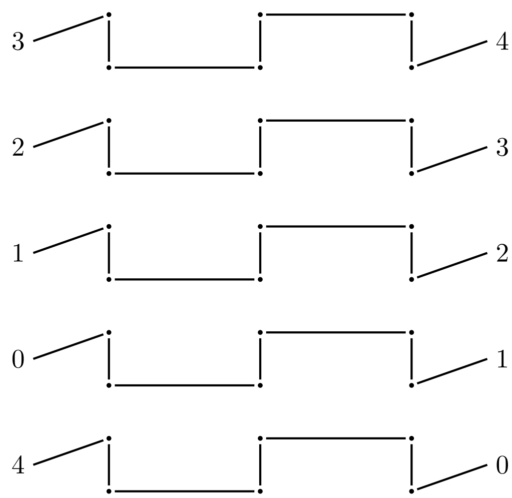

Figure 11: A Hamilton cycle in HTG(3, 10, 3)

This completes the case when $m > 1$. Now to prove that when $m = 1$ we continue to collect all desired cycles.

To prove that when $m = 1$ we have that HTG$(1, n, \ell)$ contains all cycles of length $r \equiv 2 \pmod 4$ we shall define a process called the *horseshoe method*.

Definition 8. Take the 6-path $\{0, 1, 2, \ell+2, \ell+1, \ell, 0\}$ and delete the edge between $\ell+2$ and $\ell+1$. Traverse the jump edge from $\ell+1$ to $2\ell+1$, then move to $2\ell+3$ via vertical edges. Now take the jump edge from $2\ell+3$ to 2ℓ and work your way back to the zero vertex via jump or vertical edges. This is known as the *horseshoe method*.

Theorem 3. *Honeycomb toroidal graphs with a single column contain all cycles of length* $r \equiv 2 \pmod 4$.

Proof. Take any single column of n vertices and draw in the 6-path $\{0, 1, 2, \ell+2, \ell+1, \ell, 0\}$. To extend to a cycle of length $r \equiv 2 \pmod 4$, delete the edge between w and $w+1$, where w is the largest indexed vertex in the cycle so far, and perform the horseshoe method. Repetition of this will collect all cycles of length $r \equiv 2 \pmod 4$. This is illustrated in Figures 13-15. ☐

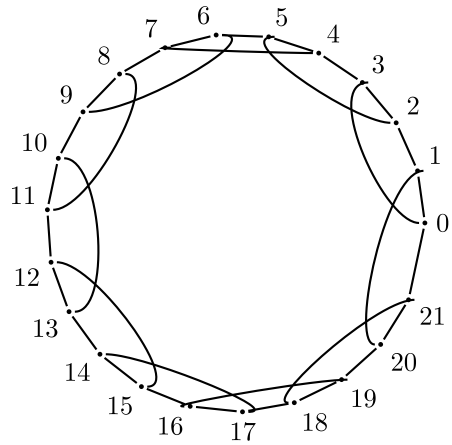

Figure 12: The complete graph HTG(1, 22, 3).

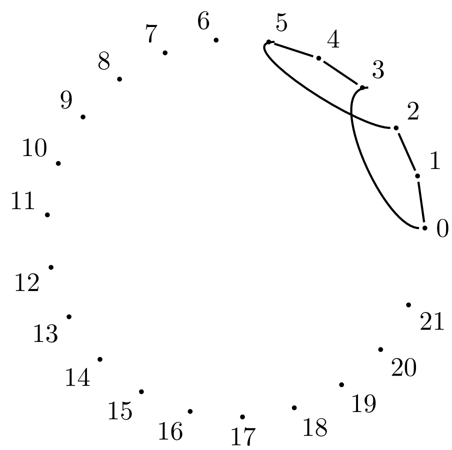

Figure 13: A 6-cycle in HTG(1, 22, 3).

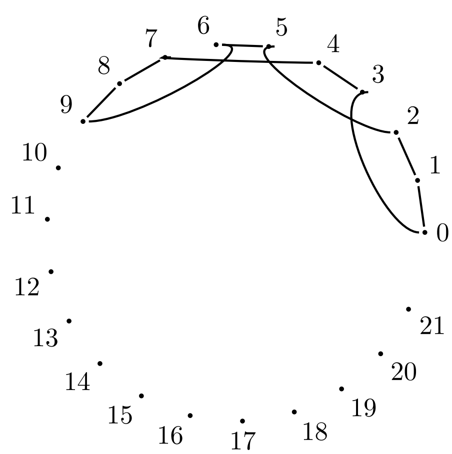

Figure 14: A 10-cycle in HTG(1, 22, 3).

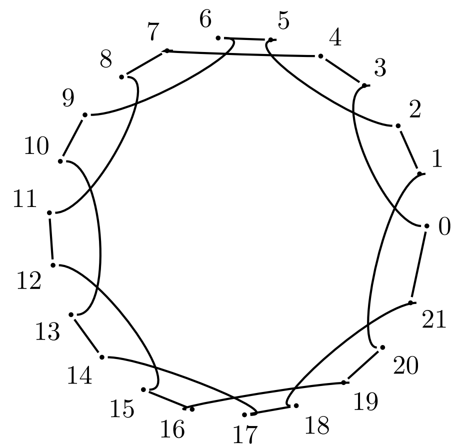

Figure 15: A Hamilton cycle in HTG$(1, 24, 3)$.

We have shown that for all m, any honeycomb toroidal graph contains all cycles of length $r \equiv 2 \pmod 4$. Now we must consider those cycles lengths in the remaining set, cycles of length $r \equiv 0 \pmod 4$.

3 Girth Results

As it turns out, these cycle lengths are not as simple as their counterparts. We can show that to have a cycle of length 4, and thus a girth of 4, the graph must exhibit one of four sufficient conditions from the following theorem:

Theorem 4. *A honeycomb toroidal graph has girth 4 if and only if it satisfies one of the following conditions:*

- $n = 4$, *or*
- $m = 1$, *and* $\ell \in \{3, n-3\}$, *or*
- $m = 1$, $n \equiv 0 \pmod 4$ *and* $\ell \in \{n/2 - 1, n/2 + 1\}$, *or*
- $m = 1$, $n \equiv 2 \pmod 4$ *and* $\ell = n/2$, *or*
- $m = 2$ *and* $\ell \in \{0, 2, n-2\}$

Proof. The $n = 4$ case is trivial as each column is an n cycle. When $m = 1$ we separate the graphs into the two subgroups $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, however the proof for each can be done together.

Consider the following construction: start at the vertex $(0, j)$, lets just call this j, we can either move vertically (up or down) or via a jump edge to begin with. Assume we jump first, that is, follow the path j to $j + \ell$. Next we can only move to either $j + \ell + 1$ or $j + \ell - 1$, so the third vertex in our path so far is $j + \ell \pm 1$.

In order for a 4-cycle to exist we must have that the following vertex is adjecent to the initial vertex j, that is, the three vertices $j + \ell$, $j + 1$, or $j - 1$. Since $j + \ell$ is already in our cycle we are left with only two penultimate positions. If we move vertically again to $j + \ell \pm 2$, it becomes clear why $\ell \in \{3, n-3\}$ is the only solution possible from here. Suppose instead we jump to $j + 2\ell \pm 1$, now it is obvious that $\ell = n/2$ is a solution when $n \equiv 2 \pmod 4$. All that is left is when $n \equiv 0 \pmod 4$ and $\ell \in \{n/2 - 1, n/2 + 1\}$. This would imply that after a second jump we are at the vertex $j + 2(n/2 \pm 1) \pm 1$, which is equivalent to $j \pm 2 \pm 1$, and has a 4-cycle for $j + 2 - 1$ and $j - 2 + 1$. Thanks to the symmetry of the honeycomb toroidal graphs, all other variations are just rotations of the argument above.

As to why these are the only cases, assume we move vertically first, to $j \pm 1$ and now jump to $j \pm 1 + \ell$, which is exactly the same as $j + \ell \pm 1$. So instead lets move twice to $j \pm 2$ first. This has already used three vertices, so the next vertex must be adjacent to j, i.e. it must be either $j+1$, $j-1$, or $j+\ell$. Clearly one of these is ruled out depending upon our initial direction of travel, and no matter which vertex we choose next we restrict ourselves to a jump of length 3. Thus, ℓ is restricted to the set $\{3, n-3\}$.

To prove that when we have two columns $m = 2$, that a graph only has girth four when $\ell \in \{0, 2, n-2\}$ we use a similar argument. Starting at any vertex (i, j) we can either move up or down, horizontally (or via a jump edge). If our vertex does not have a jump edge, then it must have a horizontal edge. Traversing this edge we arrive at $(i+1, j)$. We may now move to $(i+1, j+1)$ or $(i+1, j-1)$. The next vertex must be ajacent to (i, j), and so moving vertically will not suffice, thus we must move to $(i+1, j \pm 1 + \ell)$, and hence we see that this vertex is adjacent to (i, j) only when $\ell \in \{0, 2, n-2\}$.

If you wish to move vertically from (i, j) and then horizontally, you will always end up at either $(i+1, j+1)$ or $(i+1, j-1)$ and hence the arguments are the same. If you jump from $(i, j \pm 1)$ you end up at $(i+1, j \pm 1 \pm \ell)$ (depending on whether $i = 0$ or $i = 1$). Again we see that this only returns a 4-cycle if $\ell \in \{0, 2, n-2\}$. An example of a honeycomb toroidal graph with girth 4 is illustrated in Figure 16. □

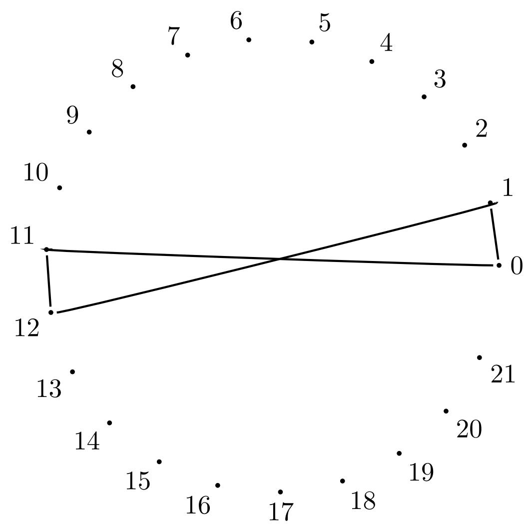

Figure 16: A 4-cycle in HTG$(1, 22, 11)$.

4 Conclusions

This concludes what we have been able to prove on the existence of cycle lengths in honeycomb toroidal graphs. An exhaustive computer search has shown us that there are graphs which frequently miss cycles of length $r \equiv 0 \pmod 4$. In particular we have found that graphs in the set HTG$(2, n, 2)$ will miss some cycles of length $r \equiv 0 \pmod 4$, even though they have girth 4.

References

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