On the Hamiltonian Laceability of Honeycomb Toroidal Graphs

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ABSTRACT

The Honeycomb toroidal graph is a highly symmetric, vertex-transitive, bipartite graph which has been investigated for certain properties including pan-cyclicity and Hamilton laceability. The main focus of this project was to construct generalised methods for finding Hamilton paths and thus provide a proof of Hamilton laceability for this graph. The resulting proof was successful for a subset of these graphs whose amount of columns is even.
1 Introduction

There has been an long interest in the existence of Hamilton paths and cycles in Honeycomb toroidal graphs. In 1981 C.C. Chen and N.F. Quimpo published a paper that showed that abelian group graphs are strongly hamiltoninan [1]. Following this in 1989 B. Alspach provided a proof that showed that every Honeycomb toroidal graph on a dihedral group has a Hamilton cycle [2]. These two papers, with their research of Cayley graphs, paved the way for interest in Honeycomb toroidal graphs. In 2015 B. Alspach expanded the knowledge of Cayley graphs by showing that some groups of them are Hamilton-laceable when bipartite and Hamilton-connected when not bipartite [3]. If the graphs studied in this paper were shown to be Hamilton-laceable then it would imply that more groups of Cayley graphs are similarly Hamilton-lacecable. These facts provided the motivation for this project.

The class of graphs denoted Honeycomb toroidal graphs are parameterised by three symbols that form the notation HTG(m, n, ℓ).

The graph is composed of a $n \times m$ array which is coordinatised as follows:

$\{(i, j) : 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}$, for $n \geq 4$, even.

The graph contains the following edges:

- $(i, j)(i, j + 1)$ for $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$,
- $(i, j)(i + 1, j)$ for $0 \leq i \leq m - 1$ where $i + j$ is odd, and
- $(m - 1, j)(0, j + ℓ)$ for $m + j$ even.

An example of the graph HTG(5, 6, 3) is shown below.

![Figure 1: Honeycomb Toroidal graph HTG(5, 6, 3)](image)

1.1 Definition. A Hamilton path in a graph $G$ is a path between two vertices $u$ and $v$ which visits every vertex of $G$ exactly once.
1.2 Definition. A graph $G$ is bipartite when the vertices are decomposed into two separate sets such that no two vertices in the same set are adjacent.

1.3 Definition. A bipartite graph $G$ is Hamilton-laceable if there exist a Hamilton path between any pair of vertices $u$ and $v$ such that $u$ and $v$ are in different parts.

1.4 Definition. A graph $G$ is vertex-transitive if there exists an automorphism which maps any given vertex to any other vertex.

The honeycomb toroidal graph is known to be bipartite and vertex-transitive and thus we need only show that there exist a Hamilton path from the vertex $(0,0)$ to any vertex $(i,j)$ where $i+j$ is odd. This ensures that we find Hamilton paths that start in the first vertex-set and terminate in the second vertex-set since the graph is bipartite.

2 Proof Setup

2.1 Definition. The starting vertex denoted $S$ is the vertex $(0,0)$ for all Hamilton paths in HTG($m,n,\ell$).

2.2 Definition. The terminal vertex denoted $T$ is a vertex $(i,j)$ where $i+j$ is odd. These vertices are where Hamilton paths in HTG($m,n,\ell$) terminate.

First we will start by finding Hamilton paths from the starting vertex $(0,0)$ to all terminal vertices $(1,k)$. To do this, we will construct generalised methods which are exemplified in Figures 2-6.

Figure 2 demonstrates the Hamilton path to the terminal vertex $T(1,4)$. If the path continued in this fashion we would have returned to $S$ and formed a cycle. Another example shown in Figure 3 contains two separate cycles starting at different vertices.
Figure 3: Connecting cycles to form a Hamilton path in HTG(2, 8, 4).

The first cycle starts at $S_1$ and skips a set of four vertices each jump it makes and returns back to $(0, 0)$. Similarly the cycle starting at $S_2$ skips the vertices contained in the first cycle. Thus $S_2$ is simply a cyclic shift of $S_1$.

Now suppose the terminal vertex $T$ of the desired Hamilton path for this graph is in the second cycle and thus we must find a way to connect both cycles such that there is a path from $S$ to $T$ containing all vertices. To do this we shall remove the edges $(1, 6)(0, 2)$ and $(0, 0)(0, 1)$ and add the edge $(0, 2)(0, 3)$. This creates a Hamilton path to $T(1, 6)$ as shown in Figure 3.

Note how the cycle that started at $S_1$ is now traversed in the opposite direction for the purpose of reaching the next cycle which terminates at $T$. This method can be used to find most Hamilton paths in $HTG\{2, n, \ell\}$ as we will show.

Before we do this it is important to look at a certain case shown in Figure 4 on the next page.
In this particular case we have the terminal vertex at (1, 8) and thus is contained in the first cycle. Therefore, we do not need the second cycle to complete the graph. Instead we utilise a vertical extension which takes each horizontal edge and raises it to the vertices below the end of each jump. By doing this all the vertices will be filled since there always exists a horizontal edge below a pair of vertices that contains jump edges. This gives us the following Hamilton path to T as shown in Figure 4.

This brings us to the following definitions.

2.3 Definition. A path $P$ in $HTG\{2, n, \ell\}$ is **vertically extendable** given the horizontal edge $(0, j)(1, j) \in P$ and $(0, j + 1), (0, j + 2), ..., (0, j + r), (1, j + 1)(1, j + 2), ... (1, j + r) \notin P$, where $(0, j + r)$ and $(1, j + r)$ are incident with a horizontal edge.

2.4 Definition. A **rung** is a 3-path generalised with the notation $R_k = (0, k)(0, k + 1)(1, k + 1)(1, k)$ for $k = 0, 2, 4, ..., n - 2$. Each rung $R_k$ is connected to a neighbouring rung $R_{k+\ell}$ via the jump edge incident with the vertices $(1, k)$ and $(0, k + \ell)$. Thus, rungs that are separated by a distance of $\ell/2$ can be cyclically joined using these jump edges. From this we obtain a 2-factor made up of $gcd(\ell/2,n/2)$ cycles of the same length. The cycles of this 2-factor are named **ladder cycles** and come equipped with the notation $C_i$ where $0 \leq i \leq gcd(\ell/2,n/2) - 1$.

2.5 Definition. An **upside down ladder cycle** noted $C'_i$ is a ladder cycle where each 3-path rung is of the form $R'_k = (0, k)(0, k - 1)(1, k - 1)(1, k)$.

2.6 Definition. Given the path $S = v_0, ..., v_k = T$ and the edge $(T, v_i)$ a **Posa exchange** transforms the path from $S$ to $v_{i+1}$ without changing the length. This is achieved by traversing the original path then taking the edge $(T, v_i)$ to the terminal vertex $T$. From here the path is traversed such that the vertex $v_{i+1}$ is reached, and thus becomes the new terminal vertex $T'$.  

Figure 4: Extending a path in $HTG(2, 8, 4)$ to a Hamilton path.
2.7 Theorem. The graph $HTG\{m, n, \ell\}$ is Hamilton-laceable for $m$ even.

Proof. To begin we need to construct a proof for $m=2$. First suppose that $T$ is the vertex with coordinates $(1, j)$ for $j \geq 2$ even. Let $b = \gcd(\ell/2, n/2)$ and $C_i$ be the ladder cycle containing the vertex $(0, 2i)$ for $i = 0, 1, ... , b - 1$. Note that $C_i$ contains the edge joining $(0, 2i)$ to $(0, 2i + 1)$.

It is obvious that $T$ belongs to some ladder cycle $C_j$ because they form a 2-factor of the graph $HTG(m, n, \ell)$. Thus employ the following procedure.

For each $i < j$, delete the edge $(0, 2i)(0, 2i + 1)$ from $C_i$ and replace it with the edge $(0, 2i + 1)(0, 2i + 2)$. This yields a path from $(0, 0)$ to $(0, 2j)$ that uses every vertex of
\[V(C_0) \cup V(C_1) \cup ... \cup V(C_{j-1}).\]

Utilising the edge $(0, 2j)(0, 2j + 1)$ we traverse $C_j$ until the terminal vertex $T$ has been reached. From here the method of vertical extension can be applied to obtain a Hamilton path from $(0, 0)$ to $T$.

Note how all Hamilton paths constructed contain jump edges except for the special case of $T = (1, 0)$ since it is the vertical extension of $C_0$. For this case we have to establish a method for finding such a Hamilton path as the fore-coming methods that extend the two column graph require jump edges.

If $b = 1$ we start with the 4-path $(0, 0)(0, 1)(1, 1)(1, 2)$, From here traverse the ladder cycle $C_0$ beginning with the edge $(1, 2)(1, 3)$. Since $b = 1$ then we have $C_0$ as a Hamilton cycle. Therefore, by traversing the path as stipulated we arrive at the terminal vertex $(1, 0)$ via the jump edge $(0, \ell)$.

If $b > 1$ then we take the path as follows. Again start with the 4-path $(0, 0)(0, 1)(1, 1)(1, 2)$. From here traverse the cycle $C_1$ until the vertex $(1, 3)$ is reached. From here use the incident edge to reach the vertex $(1, 4)$. We then traverse the cycle $C_2$ until the vertex $(1, 5)$ is reached. We continue this process until we come to the cycle $C_0$. From here we traverse $C_0$ such that we reach $(1, 0)$. This path is now vertically extendible to a Hamilton path from $(0, 0)$ to $(1, 0)$.

So far we have shown that there exists a Hamilton path from $(0, 0)$ to every vertex of the form $(1, j)$, where $j$ is even. Now suppose that the terminal vertex is of the form $(0, j)$ where $j$ is odd. Given we have ladder cycles $C_i$ for $0 \leq i \leq b - 1$ then there exists a jump edge of the form $(1, j - \ell + 1)(0, j + 1)$ in $C_{i+1}$. By starting with a Hamilton path $P$ with terminal vertex $(1, j - \ell + 1)$ in the second column we utilise a Posa exchange along the jump edge from $(1, j - \ell + 1)$ to $(0, j + 1)$, deleting the edge $(0, j)(0, j + 1)$ in the process. This method obtains Hamilton paths $P'$ to any terminal vertex $(0, j)$ where $j$ is odd excluding the cases where $(0, j) \in C_0$ and $(0, j) \in C_{b-1}$.

For the case where $T$ exists in $C_0$ we start with a Hamilton path $P$ from $S$ to $(1, j - 1 - \ell)$ as constructed above. A simple Posa exchange on the jump edge $(0, j - 1)(1, j - 1 - \ell)$ will produce a path to $(0, j)$ in $C_0$.

In order to construct Hamilton paths from $S$ to a terminal vertex $(0, j)$ that resides in the ladder cycle $C_{b-1}$ we must use upside down ladder cycles. First we produce a Hamilton path $P$ from $S$ to $(1, j + 1 - \ell)$ with the appropriate rungs $R_1', R_{1+\ell}', ..., R_{j+1-\ell}'$. Now since we have the vertex $(1, j + 1 - \ell)$ of $C_0'$ then there exists a jump edge from $(1, j + 1 - \ell)$ to $(0, j + 1)$ in $C_{0}'$. Furthermore, since the graph is vertex-transitive we know that $(0, j) \in C_{0}'$ and $(0, j) \in C_{b-1}$. Thus a Posa exchange along this jump edge will produce a Hamilton path to the vertex $(0, j)$ in $C_{b-1}$.

This concludes the proof for $m = 2$. 

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Now we shall extend the number columns to be \( m > 2 \) by inserting an even number of columns in-between columns 0 and 1, to the left of column 0, and finally to the right of column 1.

Suppose we have a Hamilton path \( P \) in HTG\((m, n, \ell)\) and we wish to insert two new columns between columns indexed by 0 and 1 as shown in Figure 5.

![Figure 5: Column insertion in-between columns for HTG\(\{2, 8, 4\}\).](image)

After insertion reindex the columns 0, \( \alpha, \beta, 1 \). Now suppose that \((0, k_1)(1, k_1), (0, k_2)(1, k_2), ..., (0, k_t)(1, k_t)\) for \( 0 < k_1 < k_2 < ... < k_t \) are the horizontal edges in the path \( P \). Delete the horizontal edge \((0, k_1)(1, k_1)\) and take the path as follows:

\[
(0, k_1)(\alpha, k_1)(\alpha, k_1 - 1)(\alpha, k_1 - 2), ..., (\alpha, k_2 + 1)(\beta, k_2 + 1)(\beta, k_2 + 2), ..., (\beta, k_1)(1, k_1)
\]

Do this for each horizontal edge in \( P \) and a Hamilton path from \((0, 0)\) to \((0, j)\) or from \((0, 0)\) to \((m, j)\) for \( j \) odd will be produced. This method can be used when making an arbitrary amount of insertions between columns 0 and 1.

Now again assume we have a Hamilton path \( P \) in HTG\(\{m, n, \ell\}\) from \((0, 0)\) to \((i, j)\) for \( i > 0 \) and we want to insert two columns to the right of the column indexed by \( i \). Index these new columns as \( \alpha \) and \( \beta \) respectively. This would produce the graph HTG\((m + 2, n, \ell)\) with a desired Hamilton path from \((0, 0)\) to \((i, j)\). Now let \((i, k_1), (i, k_2), ..., (i, k_t), 0 < k_1 < k_2 < ... < k_t \) be each vertex incident with a jump edge in \( P \). To do this we first reattach each jump edge so that it reaches column \( \alpha \). We then add the following path:

\[
(i, k_1)(\alpha, k_1)(\alpha, k_1 + 1)(\alpha, k_1 + 2)...(\alpha, k_2 - 1)(\beta, k_2 - 1)(\beta, k_2 - 2)...(\beta, k_1)
\]
Create this path for every jump edge that is found in the path in $HTG\{m, n, \ell\}$ and a Hamilton path from $(0,0)$ to $(i,j)$ in $HTG(m+2,n,\ell)$ will be produced.

Finally, suppose we have a Hamilton path $P$ from $(0,0)$ to $(0,j)$ for $j$ even in $HTG(m,n,\ell)$. Again we have to consider if the last edge in $P$ is horizontal or vertical. For the vertical edge we start with $P$ and insert two columns in-between columns 0 and 1 as before. From there we add the edge $(0,j)(1,j)$. By using this new horizontal edge along with the others already present in $P$ we utilise vertical extension to reach the rest of the vertices in the graph. After doing so we have a cycle and thus need to delete the edge connecting $(1,j)$ to the vertex that lies in the column to the left that has been previously inserted. This gives us a Hamilton path from $(0,0)$ to $(2,j)$. For the second case suppose that the last edge in $P$ is a horizontal edge. That is $T$ is either $(0,1)$ or $(0,n)$. This time we shall created two new columns to the left of the column indexed by 0 and proceed by moving each jump edge so that it reaches the left most column. Let the columns inserted to the left be indexed by $\alpha$ and $\beta$ respectively. Let all the vertices in column 0 that were originally incident with a jump edge be labelled as $(0,k_1), (0,k_2), ..., (0,k_t)$ for $k_1 < k_2 < ... < k_t$. Now starting at a vertex in column $\alpha$, which is incident with a jump edge, say $(0,u_k)$, take the following paths:

$$(\alpha,0)(\alpha,1) ... (\alpha,k_1-1)(\beta,k_1-1)(\beta,0)(0,0)$$

$$(\alpha,k_1)(\alpha,k_1+1) ... (\alpha,k_2-1)(\beta,k_2-1)(\beta,k_2-2) ... (\beta,k_1)(0,k_1)$$

$$:$$

Once the columns have been re-indexed these paths will connect the remaining vertices and complete the desired Hamilton path from $(0,0)$ to $(2,j)$ in $HTG(m+2,n,\ell)$.

Thus by showing that an arbitrary amount of insertions can be made in $HTG(2,n,\ell)$ on the left of column 1, between columns 1 and 0 and to the right of column 1 whilst still maintaining a Hamilton path from $(0,0)$ to $(i,j)$ where $i+j$ is odd, we have proven that $HTG(m,n,\ell)$ is Hamilton-laceable for $m$ even.
3 Conclusions

The proof above successfully constructs and describes methods of obtaining Hamilton paths to the appropriate vertices of the graph HTG(m, n, \ell) where the amount of columns is even. Thus this specific subset of Honeycomb toroidal graphs can be said to be Hamilton-laceable. This leaves us with the inviting problem of constructing similar methods for the remaining subset of graphs whose amount of columns is odd.

By investigating the case of \( m = 3 \) it is possible that methods of a similar nature to those discussed above would yield a sufficient proof for Hamilton-laceability. Thus, it is reasonable to think that the constructed paths would exhibit cyclic properties similar to those found in ladder cycles. Once a proof for \( m = 3 \) is validated, an extension of the graph would lead to a generalised case for any odd number of columns. However, this would still leave us with the one column graph. Thus, this special case would need to be dealt with separately.

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References

