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Lie Symmetries of Differential Equations

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1 Introduction

Conformal Galilei groups and their Lie algebras are a class of nonrelativistic algebra [1]. There are many physical applications associated with these Galilei algebras, including both classical and quantum mechanics, nonrelativistic spacetime and gravity, fluid dynamics, nonrelativistic holography and electrodynamics [2, 3]. The structure of this algebra, a non-semisimple Lie algebra, is an area of active research, in particular by Aizawa and his various collaborators [2–4]. A hierarchy of invariant equations has been constructed for the case of one underlying spatial dimension ($d = 1$) and any spin value of ℓ, but no generalised set of solutions to these equations has been found.

The equations derived from this algebra were first considered by Dobrev, Doebner & Mrugalla [5] in reference to the Schrödinger algebra (corresponding to $\ell = \frac{1}{2}$). This algebra was constructed by looking at the symmetries of the Schrödinger equation with the generators representing the invariant transformations [4].

Extending this to any $\ell \in \frac{1}{2}\mathbb{N}_0$, we get the conformal Galilei algebra. Rather than constructing an algebra from the invariant transformations of an equation, we use the algebra to find the equations, and use the generators of the algebra to find solutions to these equations. The form of these solutions are noted as eigenfunctions in [6] but no explicit expressions for these solutions are presented.

In this paper, we consider the methods of [5] and attempt to construct a explicit set of solutions to these invariant equations, for any half-odd integer value of ℓ (1/2, 3/2, 5/2, ...). We consider only these values (not the integer values) as they admit a central extension to the algebra, which is necessary if we are to use the methods from [5].

This paper is organised as follows. In the next section, we provide the generators of the ℓ-conformal Galilei algebra, including specific expressions for $\ell = \frac{1}{2}$ and $\ell = \frac{3}{2}$. We then introduce the hierarchy of invariant equations in Section 3, recounting the methods used in [3]. In Section 4 we derive a general expression for solutions to these invariant equations (for one generator only) and prove them in the final section.

2 General $\ell \in \mathbb{N}_0 + \frac{1}{2}$ central extension

We begin by expressing the generators of the ℓ-conformal Galilei algebra, using the notation described in [3]. P_n represents the spatial translation; D the dilation, C the Galilean conformal translation and H the time translation [4] We introduce the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$ [4], which is analogous to the triangular decomposition of semisimple Lie algebras.

In $\mathfrak{g}_\ell^+$:

$$H = -\frac{\partial}{\partial t}, \qquad P^{(k)} = -\sum_{j=0}^{k} \binom{k}{j} t^{k-j} \frac{\partial}{\partial x_j}, \quad k = 0, \ldots, \ell - \frac{1}{2}$$

In $\mathfrak{g}_\ell^0$:

$$M = m, \qquad D = \delta - 2t\frac{\partial}{\partial t} - \sum_{j=0}^{\ell-\frac{1}{2}} 2(\ell - j) x_j \frac{\partial}{\partial x_j}$$

In $\mathfrak{g}_\ell^-$:

$$C = t\delta - t^2 \frac{\partial}{\partial t} - t \sum_{j=0}^{\ell-\frac{1}{2}} 2(\ell - j) x_j \frac{\partial}{\partial x_j} + \frac{m}{2} \left(\left(\ell + \frac{1}{2} \right)! \right)^2 x_{\ell-\frac{1}{2}}^2 - \sum_{j=0}^{\ell-\frac{3}{2}} (2\ell - j) x_j \frac{\partial}{\partial x_{j+1}}$$

$$P^{(k)} = m \sum_{j=2\ell-k}^{\ell-\frac{1}{2}} \binom{k}{2\ell - j} (-1)^{3\ell - j + \frac{1}{2}} (2\ell - j)! j! t^{k-2\ell+j} x_j - \sum_{j=0}^{\ell-\frac{1}{2}} \binom{k}{j} t^{k-j} \frac{\partial}{\partial x_j}, \quad k = \ell + \frac{1}{2}, \ldots, 2\ell$$

These generators have the following non-trivial commutation relations:

$$\begin{aligned}
[D, H] &= 2H \\
[D, C] &= -2C \\
[D, P^{(n)}] &= 2(\ell - n) P^{(n)} \\
[C, H] &= D \\
[C, P^{(n)}] &= (2\ell - n) P^{(n+1)} \\
[H, P^{(n)}] &= -n P^{(n-1)} \\
[P^{(m)}, P^{(n)}] &= \delta_{m+n, 2\ell} (-1)^{m+\ell+\frac{1}{2}} (2\ell - m)! m! M
\end{aligned}$$

Note that these relations show $[\mathfrak{g}^0, \mathfrak{g}^\pm] \subset \mathfrak{g}^\pm$, so the decomposition above is indeed equivalent to a triangular decomposition of semisimple Lie algebras. In this paper, we will consider only the lowering generators (in $\mathfrak{g}^-$) to construct solutions.

2.1 Central extension with $\ell = \frac{1}{2}$

In the (1+1)-dimensional, $\ell = \frac{1}{2}$ case, Dobrev presents the centrally extended Schrödinger algebra. Note that in [5], the generators and mass extension m are multiplied by a factor of -1. We will continue to use the expressions shown above, and we will denote $x_{\ell-\frac{1}{2}} = x$, $x_{\ell-\frac{3}{2}} = y$, etc. for later convenience

$$\begin{aligned}
&\text{In } \mathfrak{g}^+: \\
&\quad H = -\frac{\partial}{\partial t}, \qquad P^{(0)} = -\frac{\partial}{\partial x} \\
&\text{In } \mathfrak{g}^0: \\
&\quad M = m, \qquad D = \delta - 2t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} \\
&\text{In } \mathfrak{g}^-: \\
&\quad C = t\delta - t^2\frac{\partial}{\partial t} - tx\frac{\partial}{\partial x} + \frac{1}{2}mx^2 \\
&\quad P^{(1)} = mx - t\frac{\partial}{\partial x}
\end{aligned}$$

2.2 Central extension with $\ell = \frac{3}{2}$

In the $\ell = \frac{3}{2}$ case, we aquire two more generators, $P^{(2)}$ and $P^{(3)}$, which are both generators in $\mathfrak{g}^-$. $P^{(1)}$ is now a generator of $\mathfrak{g}^+$ and changes to reflect this. The generators are presented here as we explicitly consider the $\ell = \frac{3}{2}$ case in Section 4.2. Recall we have denoted $x_{\ell-\frac{1}{2}} = x_1 = x$ and $x_{\ell-\frac{3}{2}} = x_0 = y$

$$\begin{aligned}
&\text{In } \mathfrak{g}^+: \\
&\quad H = -\frac{\partial}{\partial t}, \qquad P^{(0)} = -\frac{\partial}{\partial t}, \qquad P^{(1)} = -t\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \\
&\text{In } \mathfrak{g}^0: \\
&\quad M = m, \qquad D = \delta - 2t\frac{\partial}{\partial t} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} \\
&\text{In } \mathfrak{g}^-: \\
&\quad C = t\delta - t^2\frac{\partial}{\partial t} - 3ty\frac{\partial}{\partial y} + 2mx^2 - 3y\frac{\partial}{\partial x} - tx\frac{\partial}{\partial x} \\
&\quad P^{(2)} = 2mx - t^2\frac{\partial}{\partial y} - 2t\frac{\partial}{\partial x} \\
&\quad P^{(3)} = -6my + 6mtx - t^3\frac{\partial}{\partial y} - 3t^2\frac{\partial}{\partial x}
\end{aligned}$$

Note that the $P^{(2)}$ generator involves only one spatial variable as a linear term, unlike the $P^{(3)}$ generator. This is true for the generator $P^{(\ell+\frac{1}{2})}$ for any ℓ; the generator only linearly involves the $x_{\ell-\frac{1}{2}}$ spatial variable, whereas all other $P^{(n)}$ generators linearly involve all spatial variables for $n > \ell + \frac{1}{2}$.

3 Hierarchy of invariant equations

We now briefly recount the methods of [3] to derive a hierarchy of invariant differential equations. First, begin by considering the lowest weight vectors. The lowest weight vector, $|\delta, m\rangle$ (with lowest weight δ) is annihilated by $\mathfrak{g}^-$. Consequently, we know:

$$D\,|\delta, m\rangle = -\delta\,|\delta, m\rangle\,, \qquad M\,|\delta, m\rangle = -m\,|\delta, m\rangle\,, \qquad X\,|\delta, m\rangle = 0, \quad \forall X \in \mathfrak{g}^-$$

From this, we construct a Verma module $V^{\delta,m}$ over $\mathfrak{g}$:

$$V^{\delta,m} = \left\{ H^h \prod_{j=0}^{\ell-\frac{1}{2}} (P^{(\ell-\frac{1}{2}-j)})^{k_j}\,|\delta, m\rangle \,\middle|\, h, k_0, k_0, \ldots, k_{\ell-\frac{1}{2}} \in \mathbb{Z}_{\geq 0} \right\}$$

A Verma module is irreducible if it contains no *singular vectors*- that is, a lowest weight vector with a different eigenvalue to $|\delta, m\rangle$. It was proved in [2, 3] that, if $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 = 0$ for $q \in \mathbb{Z}_{\geq 0}$, then $V^{\delta,m}$ has one singular vector given by:

$$|v_q\rangle = \left(a_\ell m H + (P^{(\ell-\frac{1}{2})})^2\right)^q |\delta, m\rangle\,, \qquad a_\ell = 2((\ell - \frac{1}{2})!)^2$$
$$D\,|v_1\rangle = (2q-\delta)\,|v_q\rangle\,, \qquad M\,|v_q\rangle = -m\,|v_q\rangle \qquad X\,|v_q\rangle = 0, \quad \forall X \in \mathfrak{g}^-$$

We can then obtain a hierarchy of invariant differential equations by replacing the generators H and $P^{(\ell-\frac{1}{2})}$ with the differential operators:

$$\left(a_\ell m \left(\frac{\partial}{\partial t} + \sum_{j=1}^{\ell-\frac{1}{2}} j x_j \frac{\partial}{\partial x_{j-1}} \right) + \frac{\partial^2}{\partial^2 x_{\ell-\frac{1}{2}}} \right)^q f(t, x_i) = 0$$

When we take $\ell = \frac{1}{2}$ to investigate the Schrödinger algebra, we can see that these invariant equations give a hierarchy of Schrödinger equations in one dimension (for appropriate negative, pure imaginary values of m):

$$\left(2m\frac{\partial}{\partial t} + \frac{\partial^2}{\partial^2 x} \right) f(t, x) = 0$$

4 Polynomial solutions

We find polynomial solutions to the above equations, $f_{a,b,\ldots}(t, x_0, \ldots, x_{\ell-\frac{1}{2}})$, by applying the basis elements $p_{a,b,\ldots} = C^a(P^{(\ell+\frac{1}{2})})^b(P^{(\ell+\frac{3}{2})})^c \ldots$ to the lowest weight vector. In this realisation (as in [5]), the lowest weight vector is represented by the polynomial function $\mathbf{1}$, so the polynomials are of the form $f_{a,b,\ldots} = p_{a,b,\ldots}\mathbf{1}$.

Dobrev considered the cases where only one generator is applied to the lowest weight vector (all other generators are raised to a power of 0) and attemped to find a general solution. In this paper, we will concern ourselves only with solutions of the form $\left(P^{(\ell+\frac{1}{2})}\right)^b \mathbf{1}$, where $a = c = \cdots = 0$ and $b \in \mathbb{Z}^+$.

Before we consider the solutions, we must first familiarise ourselves with several functions.

$$\text{Rising Pochhammer symbol:} \qquad (b)_s = (b)(b+1)(b+2)\cdots(b+s-1) = \frac{\Gamma(b+s)}{\Gamma(b)}$$

$$\text{Degenerate hypergeometric function:} \qquad {}_1F_1\,(a;b;y) = \sum_{s\in\mathbb{Z}_+} \frac{(a)_s}{s!(b)_s} y^s$$

$$\text{Alternate form:} \qquad {}_1F_1\,(-p;b;y) = \sum_{s=0}^{p} \binom{p}{s} \frac{1}{(b)_s}(-y)^s, \quad p > 0$$

$$\text{Hermite polynomials:} \qquad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

4.1 Solutions for $\ell = \frac{1}{2}$

We now consider solutions for the case $\ell = \frac{1}{2}$ (corresponding to the Schrödinger algebra), recounting the work of [5]. We take the basis elements, $p_{a,b} = C^a (P^{(1)})^b$ and consider applying only $P^{(\ell+\frac{1}{2})}$ multiple times. The first several expressions are easily found:

$$\begin{aligned}
f_{0,1} &= mx \\
f_{0,2} &= m^2x^2 - mt \\
f_{0,3} &= m^3x^3 - 3m^2tx \\
f_{0,4} &= m^4x^4 - 6m^3tx^2 + 3m^2t^2 \\
f_{0,5} &= m^5x^5 - 10m^4tx^3 + 15m^3t^2x
\end{aligned}$$

For $a = 0, b \in \mathbb{Z}^+$, we must consider two cases: Either $b = 2k$ is even, or $b = 2k+1$ is odd. We find the following expressions (which differ from the solutions in [5] by a factor of -1 in the $\frac{mx^2}{2t}$ and $2mt$ terms due to the difference in the realisations of the generators between [5] and [2, 3]):

$$\begin{aligned}
f_{0,2k} &= \left(\frac{1}{2}\right)_k (-2mt)^k \; {}_1F_1\left(-k; \frac{1}{2}; \frac{mx^2}{2t}\right) \\
f_{0,2k+1} &= \left(\frac{3}{2}\right)_k mx(-2mt)^k \; {}_1F_1\left(-k; \frac{3}{2}; \frac{mx^2}{2t}\right)
\end{aligned}$$

These solutions can instead be written in terms of Hermite polynomials (again different to [5] by a factor of -1 in both brackets):

$$f_{0,b} = \left(\frac{mt}{2}\right)^{b/2} H_b\left(\sqrt{\frac{mx^2}{2t}}\right)$$

4.2 Solutions for $\ell = \frac{3}{2}$

Now we consider the solutions constructed using $\ell = \frac{3}{2}$, taking the basis elements $p_{a,b,c} = C^a(P^{(2)})^b(P^{(3)})^c$ and using the same method as for $\ell = \frac{1}{2}$. Again, we derive the first few expressions by applying $P^{(2)}$, and we note that they are similar to the corresponding solutions found in the case $\ell = \frac{1}{2}$:

$$\begin{aligned}
f_{0,1,0} &= 2mx \\
f_{0,2,0} &= 4m^2x^2 - 4mt \\
f_{0,3,0} &= 8m^3x^3 - 24m^2tx \\
f_{0,4,0} &= 16m^4x^4 - 96m^3tx^2 + 48m^2t^2 \\
f_{0,5,0} &= 32m^5x^5 - 320m^4tx^3 + 480m^3t^2x
\end{aligned}$$

There is a factor of 2^b which changes the coefficients of each term, but otherwise the solutions are identical to the $\ell = \frac{1}{2}$ case. We can therefore use the same form of polynomials, and find that the solutions for $\ell = \frac{3}{2}$ are of the form:

$$\begin{aligned}
f_{0,2k,0} &= 2^{2k}\left(\frac{1}{2}\right)_k (-2mt)^k\, {}_1F_1\left(-k; \frac{1}{2}; \frac{mx^2}{2t}\right) \\
f_{0,2k+1,0} &= 2^{2k+1}\left(\frac{3}{2}\right)_k mx(-2mt)^k\, {}_1F_1\left(-k; \frac{3}{2}; \frac{mx^2}{2t}\right)
\end{aligned}$$

For the purpose of writing these solutions out for general $\ell \in \mathbb{N}_0 + \frac{1}{2}$, we can instead write these polynomials as:

$$\begin{aligned}
f_{0,2k,0} &= 2^{k}\left(\frac{1}{2}\right)_k (-2 \times 2mt)^k\, {}_1F_1\left(-k; \frac{1}{2}; \frac{mx^2}{2t}\right) \\
f_{0,2k+1,0} &= 2^{k+1}\left(\frac{3}{2}\right)_k mx(-2 \times 2mt)^k\, {}_1F_1\left(-k; \frac{3}{2}; \frac{mx^2}{2t}\right)
\end{aligned}$$

4.3 Solutions for general $\ell \in \mathbb{N}_0 + \frac{1}{2}$

We now wish to consider the solutions generated by $\left(P^{(\ell+\frac{1}{2})}\right)^b \mathbf{1}$ for any $\ell \in \mathbb{N}_0 + \frac{1}{2}$. First, we look at the generators tor $\ell = \frac{5}{2}$ and $\ell = \frac{7}{2}$:

$$\begin{aligned}
\ell = \frac{5}{2}: \qquad & P^{(\ell+\frac{1}{2})} = P^{(3)} = 12mx - 3t\frac{\partial}{\partial x} - 3t^2\frac{\partial}{\partial y} - t^3\frac{\partial}{\partial z} \\
\ell = \frac{7}{2}: \qquad & P^{(\ell+\frac{1}{2})} = P^{(4)} = 144mx - 4t\frac{\partial}{\partial x} - 6t^2\frac{\partial}{\partial y} - 4t^3\frac{\partial}{\partial z} - t^4\frac{\partial}{\partial w}
\end{aligned}$$

As was mentioned in Section 2, only the $x = x_{\ell-\frac{1}{2}}$ variable has a linear term, so it is the only spatial variable to contribute to the solutions. Note the two coefficients in front of the mx and the $t\frac{\partial}{\partial x}$ terms. We can write the non-trivial part of this generator as:

$$
\begin{aligned}
P^{(\ell+\frac{1}{2})} &= \alpha_\ell mx - \beta_\ell t\frac{\partial}{\partial x} \\
\alpha_\ell &= \left(\ell+\frac{1}{2}\right)\left(\left(\ell-\frac{1}{2}\right)!\right)^2 \\
\beta_\ell &= \ell+\frac{1}{2}
\end{aligned}
$$

These coefficients, up to $\ell = \frac{7}{2}$, are:

	$\ell = \frac{1}{2}$	$\ell = \frac{3}{2}$	$\ell = \frac{5}{2}$	$\ell = \frac{7}{2}$
α_ℓ	1	2	12	144
β_ℓ	1	2	3	4

We note a connection between these coefficients and the coefficient a_ℓ from the invariant equations, namely:

$$a_\ell \beta_\ell = 2\left(\left(\ell-\frac{1}{2}\right)!\right)^2\left(\ell+\frac{1}{2}\right) = 2\alpha_\ell$$

It can be seen from the form of the generator (and by explicitly calculating the solutions) that the coefficient α_ℓ contributes to all terms proportional to the power of x, and the β_ℓ contributes to all terms proportional to the power of t. Knowing this, we can write a general expression of the solutions. The form of the solutions for general $\ell \in \mathbb{N}_0 + \frac{1}{2}$ is based off those in [5] but include these two additional coefficients:

$$
\begin{aligned}
f_{0,2k,0,\ldots} &= \alpha_\ell^k \left(\frac{1}{2}\right)(-2\beta_\ell mt)^k\ {}_1F_1\left(-k;\frac{1}{2};\frac{\alpha_\ell mx^2}{2\beta_\ell t}\right) \\
f_{0,2k+1,0,\ldots} &= \alpha_\ell^{k+1} \left(\frac{3}{2}\right) mx(-2\beta_\ell mt)^k\ {}_1F_1\left(-k;\frac{3}{2};\frac{\alpha_\ell mx^2}{2\beta_\ell t}\right)
\end{aligned}
$$

Or, representing the solutions with Hermite polynomials:

$$f_{0,b,0,\ldots} = \left(\frac{\alpha_\ell\beta_\ell mt}{2}\right)^{\frac{b}{2}} H_b\left(\sqrt{\frac{\alpha_\ell mx^2}{2\beta_\ell t}}\right)$$

Expanding these expressions, we can see these solutions are indeed the same as the cases $\ell = \frac{1}{2}$ and $\ell = \frac{3}{2}$ (and they have also been verified for $\ell = \frac{5}{2}$ and $\ell = \frac{7}{2}$):

$$
\begin{aligned}
f_{0,1,0,\ldots} &= \alpha_\ell mx \\
f_{0,2,0,\ldots} &= \alpha_\ell^2 m^2x^2 - \alpha_\ell \beta mt \\
f_{0,3,0,\ldots} &= \alpha_\ell^3 m^3x^3 - 3\alpha_\ell^2\beta_\ell m^2tx \\
f_{0,4,0,\ldots} &= \alpha_\ell^4 m^4x^4 - 6\alpha_\ell^3\beta_\ell m^3tx^2 + 3\alpha_\ell^2\beta_\ell^2 m^2t^2 \\
f_{0,5,0,\ldots} &= \alpha_\ell^5 m^5x^5 - 10\alpha_\ell^4\beta_\ell m^4tx^3 + 15\alpha_\ell^3\beta_\ell^2 m^3t^2x
\end{aligned}
$$

5 Proof for solutions of the form $\left(P^{(\ell+\frac{1}{2})}\right)^b \mathbf{1}$

We need only show that the first equation in the hierarchy ($q = 1$) holds; the rest of the equations follow trivially:

$$\left(a_\ell m\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right) f_{0,b,\ldots}(t, x_0, \ldots, x_{\ell-\frac{1}{2}}) = 0$$

It is easy to show show that the first four solutions of this form satisfy the equation. Note that we have $a_\ell \beta_\ell = 2\alpha_\ell$. For convenience, we will drop the ℓ notation in the subscripts of a, α, and β.

$$\begin{aligned}
f_{0,1,\ldots} &: \left(am\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)(\alpha m x) = 0 \\
f_{0,2,\ldots} &: \left(am\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)(\alpha^2 m^2 x^2 - \alpha\beta m^2) \\
&= -a\beta\alpha m^2 + 2\alpha^2 m^2 = 0 \\
f_{0,3,\ldots} &: \left(a_\ell m\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)(\alpha^3 m^3 x^2 - 3\alpha^2\beta m^2 t x) \\
&= -3a\beta\alpha^2 m^3 x + 6\alpha^3 m^3 x = 0 \\
f_{0,3,\ldots} &: \left(a_\ell m\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)(\alpha^4 m^4 x^4 - 6\alpha^3\beta m^3 t x^2 + 3\alpha^2\beta^2 m^2 t^2) \\
&= -6a\beta\alpha^3 m^4 x^2 + 6a\beta\alpha^2\beta m^3 t + 12\alpha^4 m^4 x^2 - 12\alpha^3\beta m^3 t x = 0
\end{aligned}$$

Now we consider the case for any $b = 2k$, moving all the terms inside the sum:

$$\begin{aligned}
am\frac{\partial}{\partial t} f_{0,2k,\ldots} &= \sum_{s=0}^{k} \binom{k}{s} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s} (\alpha m)^{k+s}(k-s)(-2\beta t)^{k-s-1} x^{2s}(-2am\beta) \\
&= \sum_{s=0}^{k} \binom{k}{s} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s} (\alpha m)^{k+s}(k-s)(-2\beta t)^{k-s-1} x^{2s}(-4\alpha m) \\
&= \sum_{s=0}^{k} -4(k-s) \binom{k}{s} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s} (\alpha m)^{k+s+1}(-2\beta t)^{k-s-1} x^{2s} \\
\frac{\partial^2}{\partial x^2} f_{0,2k,\ldots} &= \sum_{s=0}^{k} \binom{k}{s} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s} (\alpha m)^{k+s}(-2\beta t)^{k-s}(2s)(2s-1) x^{2s-2} \\
&= \sum_{s=0}^{k} 4s\left(s - \frac{1}{2}\right) \binom{k}{s} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s} (\alpha m)^{k+s}(-2\beta t)^{k-s} x^{2s-2}
\end{aligned}$$

We expand the sums and examine the nth term of the t derivative and the $(n+1)$th term of the x derivative for any n such that $0 < n < k$:

$$\left(am\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}\right)f_{0,2k,\ldots}=4\sum_{s=0}^{k}\left[-(k-s)\binom{k}{s}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s}(\alpha m)^{k+s+1}(-2\beta t)^{k-s-1}x^{2s}\right.$$
$$\left.+s\left(s-\frac{1}{2}\right)\binom{k}{s}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s}(\alpha m)^{k+s}(-2\beta t)^{k-s}x^{2s-2}\right]$$
$$=\cdots-4(k-n)\binom{k}{n}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_n}(\alpha m)^{k+n+1}(-2\beta t)^{k-n-1}x^{2n}$$
$$+4(n+1)\left(n+\frac{1}{2}\right)\binom{k}{n+1}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_{n+1}}(\alpha m)^{k+n+1}(-2\beta t)^{k-n-1}x^{2n}+\ldots$$

We note that we have:

$$\binom{k}{n+1}=\frac{k-n}{n+1}\binom{k}{n}$$
$$\text{and}\quad \left(\frac{1}{2}\right)_{n+1}=\left(\frac{1}{2}\right)_n\left(n+\frac{1}{2}\right)$$

Substituting in these relations, and recalling $a\beta=2\alpha$, we can see:

$$-4(k-n)\binom{k}{n}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_n}(\alpha m)^{k+n+1}(-2\beta t)^{k-n-1}x^{2n}$$
$$+4(n+1)\left(n+\frac{1}{2}\right)\binom{k}{n+1}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_{n+1}}(\alpha m)^{k+n+1}(-2\beta t)^{k-n-1}x^{2n}=0$$
$$\Rightarrow\left(a_\ell m\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}\right)f_{0,2k,\ldots}=\left[-(k-s)\binom{k}{s}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s}(\alpha m)^{k+s+1}(-2\beta t)^{k-s-1}x^{2s}\right]_{s=k}$$
$$+s\left(s-\frac{1}{2}\right)\binom{k}{s}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_s}(\alpha m)^{k+s}(-2\beta t)^{k-s}x^{2s-2}\Bigg]_{s=0}=0$$

So the hierarchy of invariant equations holds for all solutions of the form $\left(P^{(\ell+\frac{1}{2})}\right)^b\mathbf{1}$ for even values of b. We can show it also holds for odd values of b using the same method.

The hierarchy of invariant equations is thus solved by all solutions generated by $\left(P^{(\ell+\frac{1}{2})}\right)^b\mathbf{1}$ for any integer value of b and half-odd integer value of ℓ.

6 Conclusions and further research

The main result of this paper is the set of solutions presented in Section 4.3. These are solutions to the hierarchy of invariant equations derived from the ℓ-conformal Galilei algebra (Section 3). They hold for any half-odd integer value of ℓ and were constructed only considering one underlying spatial dimension. The solutions constructed by using other generators in $\mathfrak{g}^-$ still remain to be found explicitly, and possible extensions to integer values of ℓ could also be investigated.

In addition, applications in physics could be studied in more depth. For example, possible connections to the Pais-Uhlenbeck oscillator [7] and Lienard oscillator [8] were not explored in this paper, but are topics of interest.

References

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