Flapping of Leaves and Inverted-Flags in a Steady Flow

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1 Introduction

The fluttering of flags and leaves in a steady breeze is a common sight in nature and arises as the result of elastic and aerodynamic forces within and on the leaf or flag. This behaviour can be understood by modelling the flag as a thin cantilevered elastic sheet, a model which has been investigated extensively due to its technological applications in addition to its utility in understanding natural flow phenomena. The configuration studied in previous literature [1] has the sheet clamped parallel to the direction of flow with the clamped end incident the oncoming flow. This configuration is representative of systems able to adjust to the direction of oncoming flow such as flags on flag poles.

Recent experiments [2, 3, 4] have shown that reversing the direction of the cantilevered sheet so that the free end is incident to the oncoming flow results in flapping behaviour over a finite band of flow speeds shown in figure 1. This is in sharp contrast to the traditional flag configuration where a fluttering instability is observed over a critical flow speed. The large amplitude flapping behaviour of this inverted configuration has been shown to be a vortex-induced vibration [3] with the flapping motion of the elastic sheet driven by vortices leaving the sheet surface. It has also been shown that the band of flow speeds over which flapping occurs is dependent on sheet geometry. Figure 2 shows that flapping begins at progressively higher flow speeds as the height of the sheet is decreased while the onset of the deflected state of the sheet is insensitive to changes in sheet height. It can be seen that slender flags flap over a much smaller band of flow speeds with the flapping behaviour eventually disappearing altogether as very slender flags transition directly from undeflected to deflected states.

The relationship between the flow speed at which flapping starts and sheet geometry can be explored by analysing the stability of the undeflected state of the sheet. As the flow speed increases the aerodynamic forces on the sheet increase and compete with the elastic restoring forces of the material. Above a critical flow speed the undeflected state of the flag becomes unstable and small perturbations away from the state of equilibrium force the sheet outwards marking the beginning of the flapping behaviour. The stability of large aspect ratio sheets can be well explained [4] using classical wing theory which predicts a linear relationship between lift and angle of incidence to the flow [5]. In this case small deviations from the undeflected state result in a linear increase of the aerodynamic force. Thus as flow speed increases the sheet becomes linearly unstable, also referred to as a divergence instability.

The stability of low aspect ratio sheets however is not well explained by classical wing theory. This disagreement can be attributed to the phenomenon of ‘vortex lift’ which occurs on low aspect ratio wings and which is not modelled by traditional wing theories. Vortex lift has been shown by both experiment [6, 7] and analytic treatment [8, 9] to result in a non-linear dependence of lift on the angle of incidence of a wing. By considering the case of an infinitely slender or zero aspect ratio sheet treated by Bollay [8] Sader et. al. [4] has predicted the stability of smaller aspect ratio wings yielding good agreement with experimental results.

However the effect of sheet geometry on the dynamics and stability of the inverted flag is yet to rigorously calculated for arbitrary aspect ratio. Our aim shall be to extend the classical theory developed by Bollay [8] to wings which have arbitrary curvature along their length. This will allow the stability of sheets of arbitrary aspect ratio to be calculated and will provide insight into the
transition between the dynamics of low and high aspect ratio sheets. We shall also investigate some fundamental assumptions of Bollay’s theoretical model and explore their impact upon the case of infinitely slender wings.

Figure 1: Stroboscopic measurements of an inverted flag configuration with increasing flow speed shown left to right. The flow is incident on the left hand edge of the sheet. These images show the abrupt transition of the sheet from the straight undeflected mode to flapping mode at a critical flow speed. Also shown is the final transition to a deflected mode sheet mode as flow speed increases. Figure reproduced from [2, 3] where measurement details are provided.
Figure 2: Stroboscopic images of the sheet motion over changing sheet aspect ratio $H/L$ and flow speeds. Dimensionless flow speed $\kappa$ and corresponding air flow speed are shown along the horizontal while changing sheet height $H$ and aspect ratio $H/L$ are shown along the vertical. The stable undeflected state (red) and deflected state (blue) occur at low and high flows speed with large-amplitude flapping (green) occurring at intermediate flow speeds. Figure is taken from [3].
2 Wing Theory of Bollay

Bollay’s model for thin flat rectangular wings \[8, 9\] captures the non-linear lift of small aspect ratio wings. This effect cannot be explained using wing theories such as Prandtl’s lifting-line theory since these predict a linear relationship between lift and angle of attack. These theories also greatly underestimate the lift of small aspect ratio wings.

Bollay attempts to explain this non-linear lift relationship by modifying some assumptions of lifting-line theory. Following the same theoretical framework as Prandtl and his colleagues, Bollay approaches the problem of a wing in a uniform inviscid flow by replacing the wing surface with a distribution of bound horseshoe vortices along the chord of the wing. A horseshoe vortex is shown in figure 3 with the ‘bound’ segment running along across the span of the wing and the ‘trailing’ segments leaving at the wingtips.

Bollay can solve approximately for the flow around the wing by requiring that the combination of the free-stream flow and induced flow due to the horseshoe vortices satisfies the ‘no penetration’ boundary condition of the original flow problem and varying the strength of the horseshoe vortices accordingly.

In traditional lifting-line theory the ‘trailing’ segments of the horseshoe vortices leave parallel to the trailing edge of the wing. In a departure from the traditional method, Bollay assumes that the ‘trailing’ segments of the horseshoe vortices leave at an angle \(\theta\) to the surface of wing shown in figure 4. This introduces a non-linear relationship between the strength of the horseshoe vortices and the ‘no penetration’ boundary condition which must be satisfied at the surface. In this way Bollay introduces the non-linear lift of the low aspect ratio wing since the total strength of the horseshoe vortices can be related to wing lift using the Kutta-Joukowski theorem.

Finally, Bollay is able to determine the angle \(\theta\) of ‘trailing’ vortex segments by using Helmholtz’ laws of vortex motion \[10\] which apply to inviscid flows. These laws require that the ‘trailing’ vortices follow the direction of the local fluid flow and can thus be used to formulate a condition on the angle \(\theta\).

Bollay’s model for the case of arbitrary aspect ratio \(k\) results in the complex integral equation of the first kind

\[
\frac{k}{2\pi} \int_{-1}^{1} \gamma(\xi) \left[ \frac{1}{x-\xi} \frac{1}{\sqrt{k^2 + (x-\xi)^2}} + \frac{\cos \theta}{k^2 + (x-\xi)^2 \sin^2 \theta} \left( \frac{x - \xi \cos \theta}{\sqrt{k^2 + (x-\xi)^2}} + 1 \right) \right] d\xi = U \sin \alpha
\]

over \(x \in [-1, 1]\) where \(\gamma(\xi)\) is the unknown function governing the strength of the horseshoe vortices and \(U\) and \(\alpha\) are the free stream velocity and angle of attack respectively. The value of \(\theta\) is governed by a similar expression related to the angle of the local flow at the surface of the wing making \(\theta\) dependent on the behaviour of \(\gamma(\xi)\). Thus the integral equation is implicitly nonlinear, though approximate solutions can be wrought from equation (1) using clever approximations.

\[\text{Prandtl’s Lifting-line theory is extremely successful at predicting the lift distribution over the span of large aspect ratio wings and remains a starting point in modern wing design despite the advent of more sophisticated numerical methods.}\]

\[\text{A ‘no penetration’ boundary condition requires that fluid flow have no component in the direction normal to a solid boundary. This is so that no flow ‘penetrates’ the surface.}\]

\[\text{Not shown here for brevity}\]
These approximate solutions are shown to produce lift in agreement with experimental data \[6, 7\] for wings of aspect ratio less than unity.

### 3 Angle of Trailing Vortices

In Bollay’s analysis, when the angle of the trailing vortices \( \theta \) is determined, an interesting contradiction arises which we shall investigate here. To simplify the discussion we shall explore this contradiction in the zero aspect ratio case. Bollay’s treatment of the zero aspect ratio wing undegirds his later analysis of the more complicated finite aspect ratio case, thus by investigating a modification of this simplified case of zero aspect ratio we can, by extension, investigate some of the assumptions used in Bollay’s theory for finite aspect ratio wings. Our interest is also motivated by the utility of the zero aspect ratio case results derived by Bollay in the analysis by Sader et. al. \[4\] of slender inverted flag dynamics.

In treating the case of zero aspect ratio Bollay considered the limiting case of a wing of infinite chord and finite span \( b \) where the strength of horseshoe vortices \( \gamma(x) = \gamma_0 \) no longer varies along the chord. Sketches of the infinite wing and distribution of horseshoe vortices are shown in figures \[3, 4\].

Enforcing the ‘no penetration’ boundary condition at the wing surface and by requiring the trailing vortices to follow the direction of local flow \[4\] Bollay derives the following condition for the angle of the trailing vortices

\[
\tan(\theta) = -\frac{\cot(\alpha)}{2} + \frac{\cot(\alpha)}{2} \sqrt{1 + 2\tan^2(\alpha) \left[ 1 - \frac{2}{\pi} \arctan\left(\frac{z}{b}\right) \right]} \tag{2}
\]

where \( \alpha, z \) and \( b \) are respectively; the angle of attack, distance above the plane of the wing and span of the wing. The dependence of \( \theta \) on the distance above the wing \( z \) indicates that the trailing vortices should follow curved lines, in contradiction to Bollay’s model of straight lines of trailing vortices. Bollay chooses to relax the condition that the trailing vortices follow the local flow to resolve the ambiguity in the value of \( \theta \) and enforces the condition only at the surface of the wing \( z = 0 \). This choice of relaxation yields a normal force coefficient of

\[
C_N = 2 \sin^2 \alpha. \tag{3}
\]

It is of interest to investigate the effect of enforcing the condition given by Eq.\[2\] at different distances above the plate of the wing \( z \). By leaving the angle of the trailing vortices \( \theta \) as a function of \( z/b \) we can elucidate the dependence of the normal force coefficient \( C_N \) on the position the angle condition is enforced,

\[
C_N = 2 \left[ 1 - \frac{2}{\pi} \arctan\left(\frac{z}{b}\right) \right] \sin^2(\alpha), \tag{4}
\]

Figure \[6\] shows how the coefficient of \( \sin^2(\alpha) \) in Eq.\[4\] varies as a function of the position condition Eq.\[2\] is enforced.

#### 3.1 Extension

This contradiction has motivated the author to develop a formulation of Bollay’s theory which accounts for curved vortex lines in the zero aspect ratio case. Contrasted with the model of Bollay

\[\text{Footnote: The condition is a consequence of Helmholtz’s laws of vortex motion}\]
in Figure 5 the model allows the local angle $\theta(z)$ of the vortex lines to vary as a function of distance from the plate $z$. This new model produces a flow field dependent on the unknown strength of the horseshoe vortices $\gamma$ and the unknown function of trailing vortex angle $\theta(z)$. It can be shown that the new model’s flow field reduces to that of Bollay’s model in the case $\theta(z) = \theta_0$.

By enforcing the condition of ‘no penetration’ at the surface of the wing on our new flow field we arrive at the condition

$$2\pi \sin \alpha = \gamma \int_0^\infty \frac{1}{1 + \eta^2} \left[ \cot \theta((1-y)\eta) + \cot \theta((1+y)\eta) \right] d\eta, \quad y \in [-1, 1]$$

on $\gamma$ and the local angle $\theta(z)$. By requiring that trailing vortices follow the direction of the local flow we find that

$$(\cos \alpha + \gamma/2) \tan \theta(z) = \sin \alpha - \frac{\gamma}{\pi} \int_0^\infty \frac{\cot \theta(\xi)}{\frac{1}{2^2} + (\frac{1}{2^2} + (z-\xi)^2)} d\xi, \quad z \in [0, \infty).$$

Together Eq.5 and Eq.6 form a set of coupled non-linear integral equations with $\gamma$ and $\theta(z)$ as the unknowns.

Analysis reveals that the first integral equation Eq.5 can only be satisfied exactly when the local angle $\theta(z) = \theta_0$ is equal to a constant. Thus we can see that Bollay is only able exactly satisfy the ‘no penetration’ boundary condition by using ‘trailing’ vortices which form straight lines. Due to the vanishingly small span of the wing relative to the chord it is appropriate to only satisfy the ‘no penetration’ boundary condition along the centre line of the wing $y = 0$. This relaxation of the conditions of the first integral equation Eq.5 yields the expression

$$\pi \sin \alpha = \gamma \int_0^\infty \frac{\cot \theta(\eta)}{1 + \eta^2} d\eta.$$  

Using Eq.7 we can eliminate $\gamma$ from Eq.6 to obtain an non-linear integral equation in terms of $\theta(z)$ only

$$\frac{\pi}{2} \tan^2 \alpha = \kappa(z) \int_0^\infty \left[ \frac{1}{1 + \xi^2} - \frac{1}{4 + (z-\xi)^2} \right] \kappa(\xi) d\xi + \int_0^\infty \frac{1}{1 + \xi^2} \kappa(\xi) d\xi$$

where $\kappa(z) = \tan \alpha / \tan \theta(z)$

### 3.2 Methods of solution for the local angle

A number of different approaches for the solution of Eq.8 are described in literature [11, 12] though some of these methods must be adapted to handle the non-linear nature of the problem. The first and simplest method explored was an ‘ansatz’ solution where the functional form of $\kappa(z)$ was guessed at and arbitrary constants within the ‘guess’ were adjusted so as to satisfy Eq.8 over $z \in [0, \infty)$. Though several different ‘guesses’ were attempted none were able to approximate a solution to Eq.8.

Another method discussed extensively in the literature of integral equations is the method of solution by ‘iteration’. In this method the form of the integral equation is manipulated to produce a iteration functional, which is used to produce a sequence of improving solutions to the integral equation starting from an initial guess. Suppose for example we have an integral equation for the unknown function $f(x)$ where $x \in [a, b]$ and the integral equation can be rearranged so that

$$f(x) = G[f](x)$$
where $G$ is a functional often involving integration of $f(x)$ with some kernel function over $x \in [a, b]$. Taking an initial guess $f_0(x)$ for the solution $f(x)$ we can produce the sequence of functions

$$
\begin{align*}
    f_1(x) &= G[f_0](x), \\
    f_2(x) &= G[f_1](x), \\
    &\vdots \\
    f_{n+1}(x) &= G[f_n](x),
\end{align*}
$$

which under certain conditions can be shown to approach the true solution $f(x)$.

The form of Eq.8 can be rearranged to give an iteration functional given by

$$
h(z) = \frac{I_2[h] - I_1[h](z)}{I_2[h] \cot(\alpha) - \pi/2}
$$

where

$$
\begin{align*}
    h(z) &= \tan \theta(z), \\
    I_1[h](z) &= \int_0^\infty \frac{1}{(4 + (z - \xi)^2) \cdot h(\xi)} d\xi, \\
    I_2[h] &= \int_0^\infty \frac{1}{(1 + \xi^2) \cdot h(\xi)} d\xi.
\end{align*}
$$

By defining the unknown function $h(z)$ at collocation points along a truncated interval and using numerical quadrature [13, 14] an iteration scheme can be developed. A sequence of iterations is shown in figure 7 and demonstrates the instability of the iteration scheme above a certain number of iterations. This instability might be due to the truncated interval used in implementing the iteration scheme or may be due to the properties of the integral equation itself.

### 3.3 Conclusion

At present the author has been unable to find solutions to the non-linear integral equation Eq.8 which determines the local angle of the trailing vortices $\theta(z)$. The difficulty is in part due to the non-linear nature of the equation which introduces significant instabilities into the numerical methods explored for solution.
Figure 3: Model of zero aspect ratio wing of infinite chord and finite span $b$ from Bollay’s analysis. A single horseshoe vortex of strength $\gamma_0$ is shown with trailing segments leaving the plate at an angle $\theta$. A continuum of horse shoe vortices (not shown) is placed along the chord of the wing.

Figure 4: Model of zero aspect ratio wing (solid line) from Bollay’s analysis shown from the side. Free stream flow (long dashed line) is shown incident on wing at angle of attack $\alpha$, trailing segments of the continuum of horseshoe vortices along the chord of wing are depicted leaving the surface of wing at angle $\theta$. 
Figure 5: Bollay's zero aspect ratio model (left) is contrasted with an extended theory which allows for curvature of the trailing vortex lines (right). In the latter model the local angle of the trailing vortex lines $\theta(z)$ is a function of vertical distance from the wing $z$.

Figure 6: Variation in the coefficient of $\sin^2 \alpha$ in the expression for the normal force $C_N$ as a function of the position the angle condition is enforced $z/b$. 
Figure 7: Plots show a sequence of function \((h_0(z), h_1(z), \ldots, h_{20}(z))\) produced by the iteration scheme for the function \(h(z) = \tan \theta(z)\). The initial guess shown in the top figure (label 0) is shown to decay rapidly after which oscillations in the sequence of functions grows indicating that the iteration process is unstable. The constants labelling the figure \(L\), \(n\), \(\alpha\) and ‘relax’ denote respectively the length of the truncated interval, the number of collocation points, the angle of attack of the wing and a relaxation parameter of the iteration procedure (see [15]).
4 Extension of Finite Aspect Ratio Case

The extension of Bollay’s non-linear lift theory to a sheet with curvature along the chord can be undertaken by replacing the curved sheet with a distribution of horseshoe vortices along the chord as in Bollay’s analysis for a flat sheet. Preliminary calculations for the velocity field are being carried out. This will provide a method for calculating the aerodynamic force on a sheet of arbitrary curvature which accounts for the effect of vortex lift.

The stability of the elastic sheet or flag in a uniform flow can then be analysed by utilising a spectral method. In this method we decompose the deflection of the sheet in terms of the eigenfunctions of the elastic restoring force. The aerodynamic force can then be computed for each deflection eigenfunction using the extended method of Bollay above. The equation of motion of the sheet can then, by exploiting the orthogonality of the deflection eigenfunctions, be turned into a system of equations for the coefficients of the eigenfunctions. This set of equations can then be solved to find stable deflections of the sheet where the aerodynamic and elastic forces are balance at each point along the sheet.

Many different avenues of investigation are available from exploring how the dynamics of the inverted flapping flag change as the flag is moved closer to a wall to the interaction of two nearby flapping flags. Also of interest is the vast amount of literature on ‘vortex lift’ phenomena and extensions of Bollay’s method which could be used to develop more accurate results for the aerodynamic forces over a curved sheet.

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References


