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MATHEMATICS

Aggregation functions: properties and their visualizations

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Aggregation functions (or aggregation operators) have particular application to mathematical problems including: multi-criteria decision making; connectives in fuzzy logic; group decision making; and rule based systems. Multiple numerical inputs, usually interpreted as satisfaction of criteria, degrees of membership, support of hypotheses and so on, are aggregated to give a single numerical output or result. We might consider astronaut candidates, evaluated against multiple or *n*-criteria; or the overall satisfaction resulting from an environmentally significant decision that affects *n*-individuals with differing views.

The inputs are typically interpreted on the domain of the *n*-ary unit cube $[0,1]^n$ and the outputs on the interval [0,1]. Null scores are considered to denote no satisfaction, zero degrees of membership and so on, while scores of 1 indicate that the criterion has been completely satisfied. This leads to the first general property of aggregation functions – the boundary conditions:

$$f(0,0,\ldots,0) = 0 \text{ and } f(1,1,\ldots,1) = 1.$$

The second general property of aggregation functions is monotonicity. This property ensures that an increase in one of the inputs will not result in a decrease in output.

$$\text{i.e. } \mathbf{x} < \mathbf{y} \text{ implies } f(\mathbf{x}) \leq f(\mathrm{y})$$
$$\text{where } \mathbf{x} = (x_1,x_2,\ldots,x_n),\ \mathbf{y} = (y_1,y_2,\ldots,y_n) \text{ and } x_i < y_i \text{ for at least one } i.$$

Other properties, such as associativity, commutativity (or symmetry), idempotency etc. can be used to define different classes of aggregation operators and can also be interpreted practically toward the specific application.

e.g. Triangular norms are associative, commutative and have the further boundary conditions:

$$f(x,1,1,\ldots,1) = x \text{ and } \mathrm{f}(x,0,0,\ldots,0) = 0 \text{ for } x \text{ in any position}$$

Aggregation functions are grouped into four classes: averaging, conjunctive, disjunctive or mixed. (For 3D plots of the 2-variate cases see Fig. 1 – 4)

- *Averaging functions* (including commonly used means and weighted means) give outputs that lie between the greatest and smallest elements of input.
- *Conjunctive functions* simulate the logical “AND” and give outputs no greater than the smallest element of input.
- *Disjunctive functions* are dual to conjunctive functions and simulate the logical “OR”. They give outputs no smaller than the largest element of input.
- *Mixed aggregation* functions exhibit different behavior (conjunctive, disjunctive or averaging) depending on the region of input.

The purpose of this particular study was to produce 3-D graphs for a number of aggregation operators in the 2-variate case. To do this required a thorough understanding of the equations used to generate the plots and of the general aggregation properties. Seeing the finished graphs also greatly enhanced my

Simon received an ICE-EM Vacation Scholarship in December 2006.
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comprehension of the topic. There are a number of uses for aggregation functions in areas as diverse as economics, politics, medicine and logic.

I found the AMSI vacation scholarship to be a rewarding and challenging experience. I especially enjoyed contributing to the work of my supervisor, Gleb Beliakov, and am motivated to begin my Honours year with this invaluable research behind me.

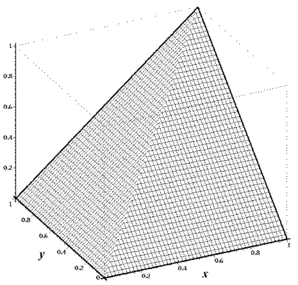	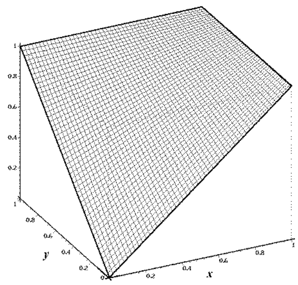
Figure 1. minimum – (conjunctive) $T(x,y) = \min(x,y)$	**Figure 2. probabilistic sum – (disjunctive)** $S(x,y) = x + y - xy$
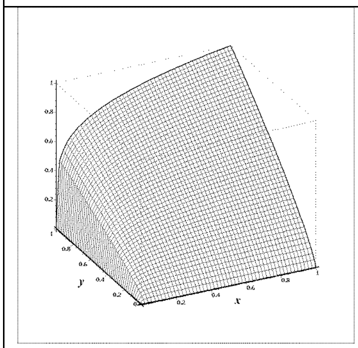	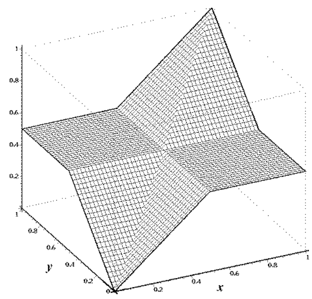
Figure 3. weighted geometric mean – (averaging) **M(*x*,*y*) =** [illegible]	**Figure 4. a median (also the nullnorm) – (mixed)** $N(x,y) = \text{med}(x,y,0.5)$

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