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Topology and the Cantor Set

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The initial intent of my work was to investigate some of the properties of the Cantor Set. I soon realised that I would need to know some general topology, so I borrowed a text called 'Topology for Analysis' by Albert Wilansky. I found this book to be very good and highly recommend it. I spent most of my time working through the first several chapters. Some concepts that were new to me were those of bases, filters, nets and several separation axioms.

The Cantor Set, C, can be characterised as being zero-dimensional, perfect and compact. These are defined, for a set X, as:

1. Zero-dimensional: X has a base of clopen sets
2. Perfect: $X = X'$ where X' equals the set of limit points of X
3. Compact: every open cover of X has a finite subcover.

To construct a clopen base for C, take the set of all intervals that occur in the construction of the set, i.e. [0,1], [0,1/3], [2/3,1], [0,1/9], [2/9,1/3],... If we intersect each of these with C, the collection of all these intersections forms a clopen base. This base is also countable, so we have that C is second countable. Zero-dimensional also implies totally disconnected. Since C is perfect, we have that totally disconnected does not imply the discrete topology, as I had initially thought. Q with the relative topology is also another counterexample.

It can be shown that any set with these properties is homeomorphic to C. This fact can be used to show the existence of a closed, uncountable set of irrationals. We note that $C \times C$, is also zero-dimensional, perfect and compact, hence homeomorphic to C, and $\{c\} \times C$, $c \in C$, is a disjoint family of homeomorphs of C, one of which doesn't contain any rationals.

The text discussed convergence in terms of nets and filters, these are defined as:

1. Net - a net is a function defined on a directed. a directed set, X, being a partially ordered set with the additional condition that for each $x, y \in X$, there exists $z \in X$ with $z \geq x$, $z \geq y$.

2. Filter - a filter, F, in X is a collection of subsets of a set X if
   - (i) $X \in F$;
   - (ii) $\phi \in F$;
   - (iii) $A \in F$, $B \in F$ implies $A \cap B \in F$
   - (iv) $A \in F$, $B \supset A$ implies $B \in F$

The need for nets and filters was motivated by the result that in a space X that isn't first countable if $A \subset X$ then $y \subset$ □ does not imply there exists $y_n \in A$ such that $\{ y_n \} \to y$. Sequences are a type of net.

The separation axioms for T_0, T_1, T_2, T_3, $T_{3.5}$, T_4, regular and normal spaces were also very interesting.

Trent received an ICE-EM Vacation Scholarship in January 2005.
See www.ice-em.org.au/students.html#scholarship.