Functional Analysis and its Applications

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My project began with a reading course into a number of areas of functional analysis, before moving on to looking at applications of the mathematical foundations that I had familiarised myself with. Put most broadly, functional analysis concerns the study of the properties of various types of spaces and the operators on them. In general, the more structure that is defined on a given space, the more one knows about the properties that it must have.

To see an example of this, we can look at the Hahn-Banach theorem, and analyse what it implies in various settings. A common version of the Hahn-Banach Theorem is formulated on a vector space, and says that a linear functional defined on a subspace of a given vector space can always be extended in such a way that it remains dominated by a sublinear functional which dominated the original linear functional. A vector space has little structure in comparison to other common spaces, and this theorem can give other important results if we move to spaces with more structure. On a topological vector space the Hahn-Banach theorem as previously stated can give a result concerning the separation of disjoint convex sets by a continuous linear functional. If the topological vector space is locally convex, even stronger results can be proved. On normed spaces, the Hahn-Banach theorem implies that every bounded functional defined on a subspace of a normed space has an extension to the entire space with the same norm as the original functional.

Looking further than these closely related results, it is found that the Hahn-Banach theorem has a wide range of diverse applications, even outside the realm of mathematics. It has been used, for example, to prove a generalised version of the fundamental theorem of asset-pricing, and in showing the existence of a universal entropy function in a classical thermodynamic setting. I studied the framework for the latter application quite thoroughly in the latter stages of my project. The basic idea of this approach is to show that a suitable entropy function exists without using any preconceived notions of heat or temperature to define such a function.

To begin with, consider a space state Γ, whose elements are all of the possible equilibrium configurations of a given amount of a given substance. Next a type of ordering relation is defined on this space state, where, for x and y in Γ, it is said that x

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precedes y if it is possible to go from state x to state y in an adiabatic process. If a few simple axioms are given to this 'precedes' relation, then it can be shown that an entropy function (denoted by S) exists on this space state that encodes all of the information of this ordering. That is, $S(x) \leq S(y)$ if and only if x precedes y, with S also satisfying additional additivity and extensivity requirements. An extra assumption that is required here (which can be shown from a number of additional axioms) is that any two states in a given state space are comparable; that is for x and y in Γ either x precedes y or y precedes x. Having shown that entropy functions exist on single space states, it is then shown that these functions can be chosen to be consistent when space states are allowed to interact with each other. Finally, processes in which state spaces are not necessarily conserved (such as chemical reactions) are considered, and it is here that the Hahn-Banach theorem is required to show that a universal entropy function which is consistent for such processes (and thus all processes in nature) does in fact exist.

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