Teletraffic Theory: Practical Loss Networks

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INTRODUCTION

Accurate models of telecommunication networks are necessary to quantitatively describe their performance under differing loads and configurations. 'Performance' in this scenario refers to the probability of arriving calls being declined (called blocked or lost) due to insufficient available network resources; these blocking probabilities are otherwise known as the Grade of Service of the network.

Modern telecommunications networks are very sophisticated, utilising such techniques as: packet-switching, dynamic alternate routing and trunk reservation. These techniques will not be explained in this document, though the model given aptly describes the original circuit-switched telephone network.

MODEL

In a circuit-switched network each call utilises a set of communication links for the duration of the call. Say that the network has J communication links, each link j consisting of C_j circuits. The quantity of circuits in each link is the network's capacity constraint and form a vector $\boldsymbol{C}$. The simplifying assumption of fixed routing is made, i.e. that there exists only one route through the network for each pair of caller-receiver. In this way, the different types of calls are represented by these routes, collected into a set $\mathcal{R}$.

To record which routes require which links, a matrix of size $J \times |\mathcal{R}|$ denoted A is constructed. In this matrix an element $a_{jr} \in \mathbb{N}_0$ stores the number of circuits required in link j for a call on route r. This general setup is further simplified by only considering networks where A is a $0-1$ matrix. Each route r is supplied offered traffic at rate ν_r, which when collected into a vector of length $|\mathcal{R}|$ is denoted $\boldsymbol{\nu}$.

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Once it is assumed that call lengths are exponentially distributed (with mean of 1) then the model is simply a birth-death process. The state of the network $\boldsymbol{n} = (n_r, r \in \mathcal{R})$ stores the number of calls in progress along each of the routes; the capacity constraints on the network limit the state space to the polytope

$$S(\boldsymbol{C}) = \{\boldsymbol{n} \in \mathbb{N}_0^{|\mathcal{R}|} : A\boldsymbol{n} \leq \boldsymbol{C}\}.$$

Thus a time-reversible Markov Chain $(\boldsymbol{n}(t), t \geq 0)$ can be analysed for properties of interest, namely the blocking probabilities. The stationary distribution is derived as

$$\pi(\boldsymbol{n}) = G(\boldsymbol{C})^{-1} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}$$

with a normalising constant

$$G(\boldsymbol{C}) = \sum_{\boldsymbol{n} \in S(\boldsymbol{C})} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}.$$

For the blocking probabilities to be computed then a definition of blocking must be given; a state $\boldsymbol{n}$ is blocking for a route r if $\boldsymbol{n} + \boldsymbol{e}_r \notin S(\boldsymbol{C})$. The likelihood of this occurring can be shown in terms of the normalising constant $G(\boldsymbol{C})$ to be

$$L_r = 1 - \frac{G(\boldsymbol{C} - A\boldsymbol{e}_r)}{G(\boldsymbol{C})}.$$

Unfortunately it is not computationally feasible to calculate these blocking probabilities exactly as computing $G(\boldsymbol{C})$ is #P-complete (Louth 1990).

ERLANG FIXED POINT APPROXIMATION

In order to approximate a network of J links, the Erlang fixed point approximation (EFPA) instead looks at J networks of individual links. The blocking probability for a single link with offered traffic ν and capacity C is given by Erlang's formula, which is

$$E(\nu, C) = \frac{\nu^C}{C!} \left[\sum_{n=0}^{C} \frac{\nu^n}{n!} \right]^{-1}.$$

The EFPA solves the following fixed point equation,

$$B_j = E(\rho_j, C_j), \quad j \in J,$$

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where

$$\rho_j = \sum_{r \in R_j} \nu_r \prod_{i \in r \setminus \{j\}} (1 - B_i), \quad j \in J.$$

Here B_j is approximately the blocking probability for any traffic along link j, and the blocking probability for an entire route is simply

$$L_r \approx 1 - \prod_{j \in r} (1 - B_j).$$

The term ρ_j is called the reduced load to link j, and is the expected rate of offered traffic for that link. The form of ρ_j and L_r embeds the assumption that links block independently to each other, so this scheme is often called the reduced load assumption. Fixed point iteration is an effective method for solving this set of equations; a unique solution must exist by Brouwer's fixed point theorem (Kelly, 1991).

TWO-LINK APPROXIMATION SCHEMES

There have been some investigations into *two-link approximation schemes* to improve upon EFPA's accuracy. These schemes break the original network into subnetworks - each containing two links - which would be solved analytically given some assumption/approximation of the incoming traffic.

Two-link schemes are designed to reduce the effects of EFPA's major flaw which is the assumption that links block independently. If a network has some links which are common to many routes then there will be a significant correlation in their link blocking probabilities. The two-link schemes try to add a level of link dependency so that this phenomenon can be modelled.

Early work by Bebbington, Pollett and Ziedens (1997) started by creating a special two-link approximation scheme for symmetric ring networks. Symmetry allowed for computationally feasible analytical solutions to be found so that the error could be measured. Figure 1 compares the error in the derived approximations to the EFPA for the routes that only contain one link (called type-1 calls) and similarly Figure 2 compares the error in the routes that contain two links (type-2 calls).

Work on a more general two-link approximation scheme was completed by Thompson (2002). In this work, Thompson created a generic approximation scheme which added substantial complexity to the fixed point equation. Thompson compared his general model against the EFPA and many special-case approximations (like the ring network above). In the cases considered the general two-link approximation outperformed EFPA, and was comparable to the special-case methods; though the extra accuracy came at the cost of much longer computation times.

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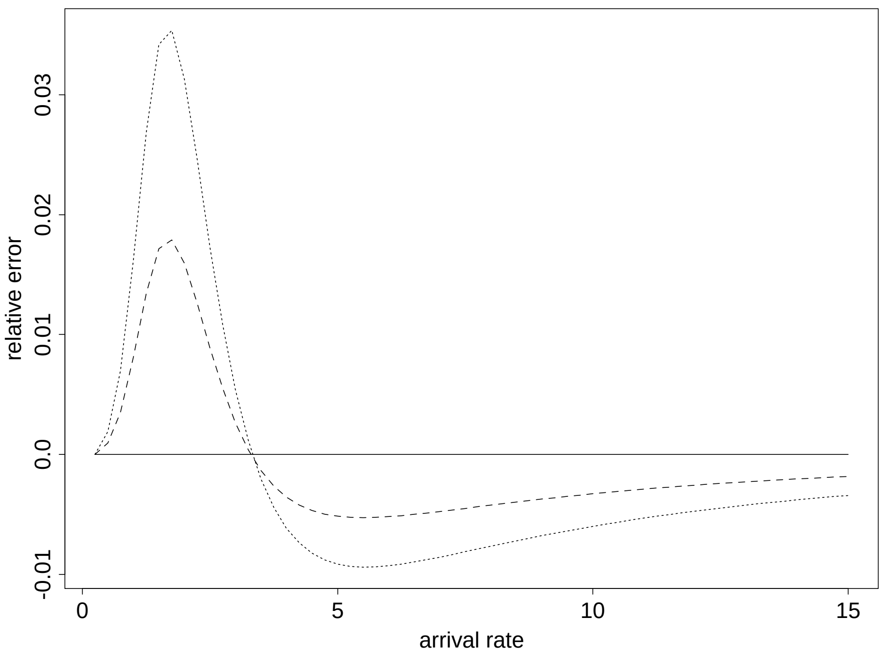

Figure 1: Relative error in the estimated blocking probability for type-1 calls
· · · · EPFA - - - - Approx. I _____ Approx. II

KELLY LIMITING REGIME

The seminal paper by Kelly (1991) proved many interesting limit theorems involving these teletraffic models and the EFPA. One in particular showed the accuracy of the EFPA under a limiting regime which has become known as the *Kelly limiting regime*.

Consider a sequence of networks indexed by N, with link capacities $C_j(N)$ and offered traffic $\nu_r(N)$, where A is fixed. In this sequence as $N \to \infty$ then ensure that $\exists\ \nu_r, C_j$ such that

$$\lim_{N\to\infty} \frac{1}{N}\nu_r(N) \to \nu_r, \quad \lim_{N\to\infty} \frac{1}{N}C_j(N) \to C_j.$$

Also impose the mild condition on supply and demand such that

$$C_j(N) - Mo(U_j) \in o(N^{1/2})$$

where $Mo(U_j)$ is the utilisation (number of circuits in use) of link j in the modal state.

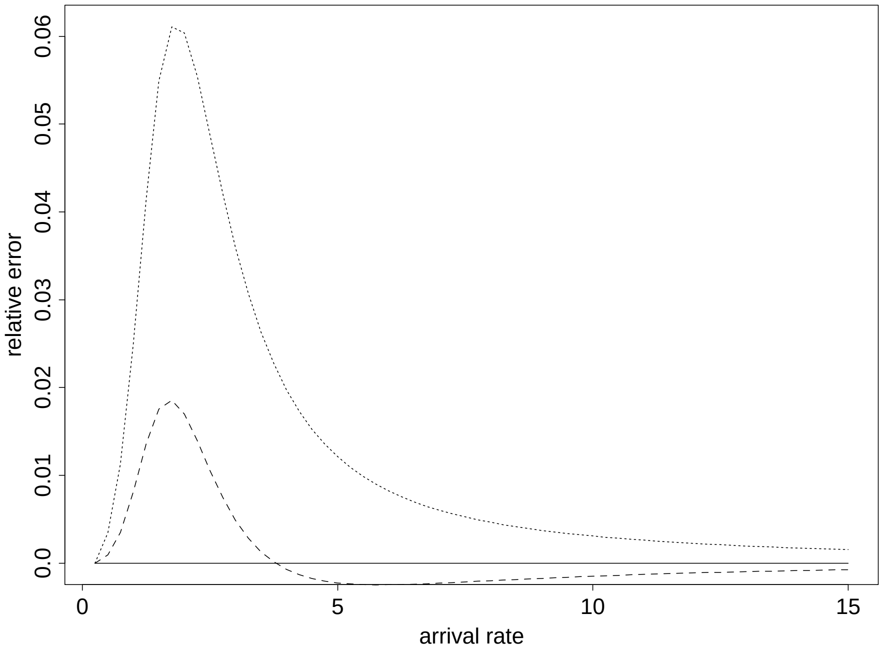

Figure 2: Relative error in the estimated blocking probability for type-2 calls
· · · · EPFA - - - - Approx. I _____ Approx. II

Lastly, assume that there are no critically loaded links, i.e. no links where

$$Mo(U_j) = C_j \text{ but } B_j = 0.$$

Under this limiting regime, the result of performing the EFPA approaches the exact solution. Therefore the computationally efficient EFPA, which relies on the flawed assumption of independently blocking links, will perform better as the network grows in supply and demand. This theorem gives a justification to teletraffic engineers to prefer the EFPA over complicated two-link schemes.

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CONCLUSION

Teletraffic theory is the field which investigates the efficiency of telecommunication networks. In particular the likelihood of calls blocking is sought after. Analytic solutions are computationally infeasible for networks of scale so approximations must be used. A fast and effective approximation is the Erlang Fixed Point approximation (EFPA) which performs better for any network as the supply and demand increases. More complicated two-link approximation schemes are also available, yet they tradeoff some small increase in accuracy with substantial extra computation time. The discussion has focused on the fixed-route circuit-switched network, where further work would incorporate techniques such as: packet-switching, dynamic alternate routing and trunk reservation.

I would like to thank Prof. Phil Pollett for being very generous with your time and knowledge. Your encouragement helped keep me motivated to investigate not only teletraffic theory, but also many fascinating side-problems like simulation techniques. Also I would like to thank AMSI for letting me participate in this great program. The Big Day In was lots of fun, stimulating, and has demystified what a research mathematics lifestyle would involve. Thanks to MASCOS for being a great audience for my practice presentation, and finally thanks to Prof. Joe Grotowski for helping organise my last-minute application.

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