Euler's Theorem

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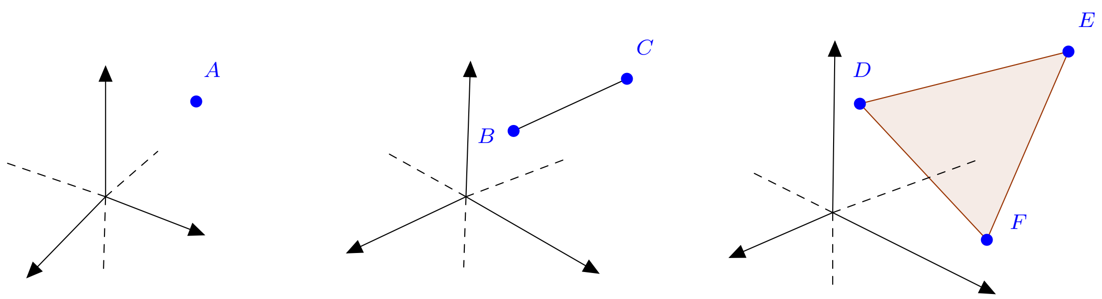
1 Introduction

This summer I have spent six weeks of my holidays working on a research project funded by the AMSI. The title of my project was Euler's Theorem, however I learnt far more than a single theorem. My project consisted of two parts: studying a topic of mathematics called simplicial homology so as to prove Euler's theorem and then applying what I had learnt to answer several research questions. In this report I shall elaborate further on what I have learnt regarding simplicial homology and discuss the aims and outcomes of my research.

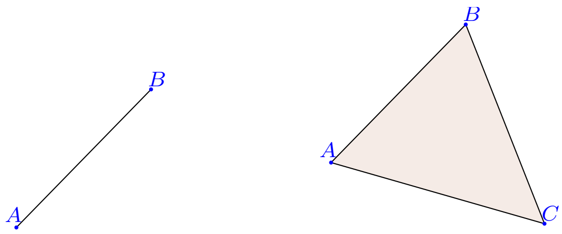
2 Preliminaries

The foundation of simplicial homology is called a *simplex*. A p-simplex is essentially the p-dimensional analog of a triangle. For instance a zero-simplex is a point, a one-simplex is a line segment and a two-simplex is a triangle and so on.



zero-simplex   one-simplex   two-simplex

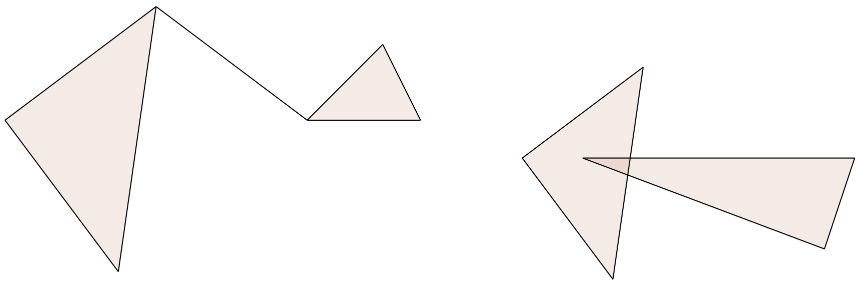
Often the notation σ^k is used to denote a simplex with $k+1$ vertices. A simplex σ^k is a *face* of a simplex σ^n, $k \leq n$, means that each vertex of σ^k is a vertex of σ^n. This fairly intuitive notion is illustrated by way of example below.



$\langle AB \rangle$ $\langle ABC \rangle$

Here the simplex $\langle AB \rangle$ is a face of the simplex $\langle ABC \rangle$.

The next most fundamental construction is called a *geometric complex* which is essentially a special collection of simplices. The complex is a 'special' collection in that it is a set of simplices satisfying two properties. The first property is that every face of a member of the set must also be within the set. The second property is that all simplices are *properly joined* , meaning that the intersection of any two simplices is either empty or a face of both.



Properly Joined Not Properly Joined

The *polyhedron associated with the complex K* is the union of the members of K with the Euclidean subspace topology. If the polyhedron associated with the complex K is homeomorphic to a topological space X then that space is said to be *triangulable*, and K is a *triangulation* of X.

A simplex can be oriented by choosing an ordering for the vertices. For example:

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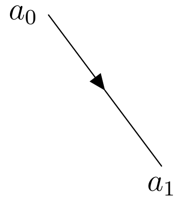
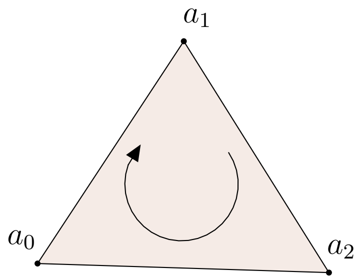


Ordering: $a_0 < a_1$



Ordering: $a_0 < a_1 < a_2$

The equivalence class of even permutations of the ordering of vertices determines determines the positively oriented simplex, $+\sigma^n$, whilst the equivalence class of odd permutations determines the negatively oriented simplex, $-\sigma^n$. A complex is said to be oriented if each of it's simplices is oriented.

Definition 2.1. Let K be an oriented geometric complex. If p is a positive integer, then a *p-dimensional chain* is a function, c_p from the family of simplices in K to the integers such that for each p simplex σ^p, $c_p(-\sigma^p) = -c_p(-\sigma^p)$. With the operation of pointwise addition the set of all p-dimensional chains forms a group called the *p-dimensional chain group.*

An *elementary p-chain* is a chain with the property that for each simplex δ^p distinct from σ^p, $c_p(\delta^n) = 0$, and $c_p(\sigma^p) = g$. The notation, $g \cdot \sigma^p$, is frequently used to denote the p-chain $c_p(+\sigma^p) = g$. An arbitrary chain may be written as a finite summation of these elementary chains as follows:

$$c_p = \sum_{i=1}^{\alpha_p} g_i \cdot \sigma_i^p$$

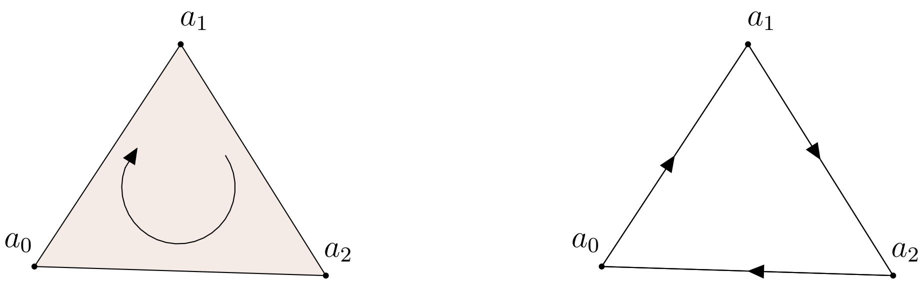
where α_p is the number of p simplices.

Given two simplices σ^p and σ^{p-1}, the incidence number, $[\sigma^p, \sigma^{p-1}]$, is equal to zero if σ^{p-1} is not a face of σ^p otherwise it is either $+1$ or -1 depending on the orientation. Label the vertices of σ^{p-1} as $a_0, ..., a_{p-1}$ and let v denote the vertex of σ^p which is not in σ^{p-1}, then if $+\sigma^p = \langle v a_0 ... a_{p-1} \rangle$ the incidence number is 1, and if $-\sigma^p = \langle v a_0 ... a_{p-1} \rangle$ the incidence number is -1.

The *boundary* of an elementary p-chain $g \cdot \sigma^p$ $(p \geq 1)$ is defined as follows:

$$\partial(g \cdot \sigma^{p+1}) = \sum [\sigma^p, \sigma_i^{p-1}] g \cdot \sigma_i^p$$

The boundary of the $1 \cdot \langle a_0 a_1 a_2 \rangle$ is $\langle a_1 a_2 \rangle$ - $\langle a_0 a_2 \rangle$ + $\langle a_0 a_1 \rangle$ (see figure below).



The boundary operator is extended by linearity to apply to general p-dimensional chains. In particular:

$$C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1}$$

It can be proven that for any $p \in \mathbb{N}$ and for any (p+1)-dimensional chain c_{p+1},

$$\partial_p \left(\partial_{p+1} \left(c_{p+1}\right)\right) = 0. \tag{1}$$

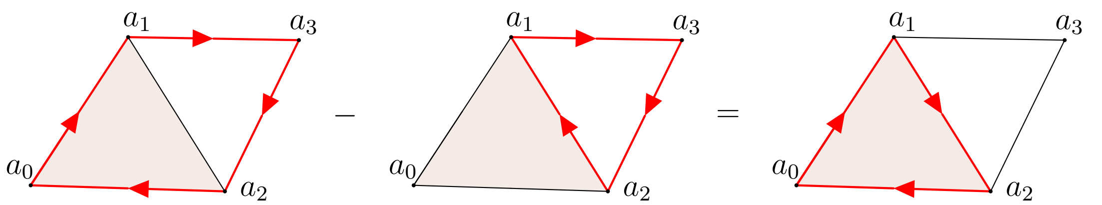
A *p-dimensional cycle* is a p-dimensional chain, z_1 such that $\partial\left(z_1\right) = 0$. A *p dimensional boundary* is a chain, b_1 which there exists a (p+1)-dimensional chain, c_{p+1}, such that $\partial\left(c_{p+1}\right) = b_1$.

Clearly every boundary is a cycle because of Equation 1, however the question: is every cycle a boundary is an important question which leads into the development of the concept of homology. The answer to the question is of course no, in particular cycles which are not boundaries are called non-bounding cycles. In homology theory the non-bounding cycles are the interesting ones as they provide a way by which we can associate an algebraic structure to a topological space.

Two p-dimensional cycles, w_1 and z_1 are said to be *homologous* to each other, $\sim$, if there exists a (p+1)-dimension chain, c_{p+1}, such that:

$$w_1 - z_1 = \partial\left(c_{p+1}\right)$$

The following is an example of two cycles that are homologous to each other:



The equivalence relations, $\sim$, partitions the group of p-dimensional cycles into cells called *homology classes.* It can be shown that the cell:

$$\{z_p \in Z_p(K) : z_p \sim w_p\}$$

is equivalent to the left coset:

$$w_p + B_p(K) = \{w_p + \partial(c_{p+1}) : c_{p+1} \in C_{p+1}(K)\}$$

The set of all such left cosets along with the appropriate binary operation is called the *p-dimensional homology group.* In particular the p-dimensional homology group is the following quotient group:

$$H_p(K) = Z_p/B_p$$

The rank of the free part of the p dimensional homology group is called the *pth betti number* and denoted $R_p(K)$.

Definition 2.2. If K is a complex of dimension n, the number

$$\chi(K) = \sum_{p=0}^{n}(-1)^p R_p(K) \tag{2}$$

is called the *Euler characteristic of K.*

If K_1 and K_2 are two triangulations of the same surface, Σ, then $R_p(K_1) = R_p(K_2)$ and so it makes sense to talk about the betti numbers of the surface Σ, $R_p(\Sigma)$ and consequently it also makes sense to talk about the Euler characteristic of a surface. The betti numbers and consequently the Euler characteristic are also topological invariants which means that if two spaces are homeomorphic to each other then they will have the same betti numbers, and Euler characteristic.

3 Euler's Theorem

Euler's theorem relates the number of edges, vertices and faces any simple polyhedra has by the equation $V - E + F = 2$. Where a simple polyhedron is just a rectilinear polyhedron (a solid in 3-space bounded by convex polygons) whose boundary is homeomorphic to a sphere.

There are several ways of proving this theorem, the one that I studied is actually a proof of a generalisation of Euler's theorem called the Euler-Poincare Theorem. Below is a statement of the Euler-Poincare theorem, for a full proof refer to 'Basic Concepts of Algebraic Topology' by Fred H. Croom.

Theorem 3.1 (The Euler-Poincare Theorem)**.** *Let K be an oriented geometric complex of dimension n, and for p = 0, 1, ..., n let α_p denote the number of p-simplices of K. Then*

$$\sum_{p=0}^{n}(-1)^p\alpha_p = \sum_{p=0}^{n}(-1)^p R_p(K)$$

where $R_p(K)$ denotes the pth betti number of K.

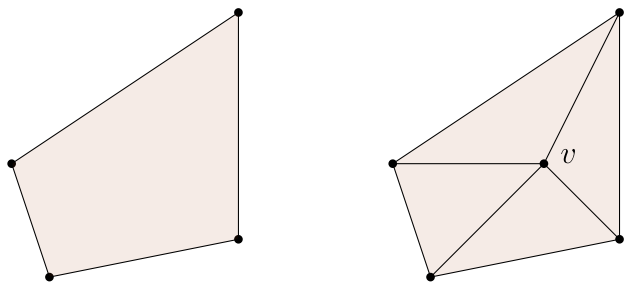
At first glance it may not be immediately obvious how this generalises Euler's theorem, and indeed there are several intermitent steps required to convince oneself that it does. The subsequent argument closely follows one produced by Croom (1978).

Consider a simple polyhedron P as specified in Euler's Theorem. If all the faces were triangular then

$$\begin{aligned} V - E + F &= \alpha_0 - \alpha_1 + \alpha_2 \\ &= 2. \end{aligned}$$

by the Euler-Poincare theorem since the Euler characteristic of a sphere is 2. However the faces of a simple polyhedron needn't be triangular.

Consider a simple polyhedra, P, with at least one non-triangular face, S. Suppose that S has n_0 vertices, n_1 edges and 1 face, then $V - E + F = n_0 - n_1 + 1$. S can be triangulated by considering a new vertex, v, in the interior of the polygon S and joining all other vertices to v with a line segment to create new faces, and edges.



During this process:

1. The number of vertices changes from n_0 to $n_0 + 1$
2. The number of edges changes from n_1 to $n_1 + n_0$
3. The number of faces changes from 1 to n_0

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Now consider the effect of this triangulation on the value of $V - E + F$.
$V - E + F = (n_0 + 1) - (n_1 + n_0) + n_0 = n_0 - n_1 + 1$.

So the triangulation process does not have any affect on the value of $V - E + F$. Suppose that by triangulating each face we obtain the polyhedron P' and a triangulation K consisting of the faces, edges and vertices of P'. Then K is a triangulation of P' and consequently a triangulation of a sphere. The final count of $V - E + F$ which is $\alpha_0 - \alpha_1 + \alpha_2$ will be the same as the initial $V - E + F$.

Finally applying the Euler-Poincare Theorem:

$$\alpha_0 - \alpha_1 + \alpha_2 = \chi(K)$$
$$\therefore V - E + F = 2.$$

4 Triangulations

4.1 The Trade Off

When triangulating any topological space there is always a trade off between the triangulation being easy to draw and the triangulation being easy to work with. The objective of this section was to investigate some triangulations which satisfy both these criterion. The main topological spaces we shall look at in this section are orientable surfaces.

If a 2-dimensional manifold is triangulable each of it's triangulations will be a 2-pseudo manifold. A *2-psuedo manifold* is a complex like any other triangulation, however with three additional conditions:

1. Each simplex of K is a face of some n-simplex of K.
2. Each 1-simplex is a face of exactly two 2-simplexes of K.
3. Given a pair σ_1^n and σ_2^n of n-simplices of K, there is a sequence of n simplices beginnning with σ_1^n and ending with σ_2^n such that any two successive terms of the sequence have a common $(n-1)$-face.

The second property in conjunction with the Euler-Poincare Theorem affords us the ability to prove the following three relationships.

Theorem 4.1. *Let K be a 2-pseudomanifold with α_0 vertices, α_1 1-simplices, and α_2 2-simplices. Then*

1. $3\alpha_2 = 2\alpha_1$

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2. $\alpha_1 = 3(\alpha_0 - \chi(K))$

3. $\alpha_0 \geq \frac{1}{2}\left(7 + \sqrt{49 - 24\chi(K)}\right)$.

The one of most importance to us in this context is the third which gives us a way of determining a tight lower bound for the number of vertices a triangulation requires.

We see that a sphere for example having Euler characteristic equal to 2 requires at least 4 vertices, 6 edges and 4 faces. This is exactly the number of vertices, edges and faces the tetrahedron has (which is a triangulation of the sphere), and so we call the tetrahedron (or more precisely the complex that the tetrahedron is associated to) the optimal triangulation of the sphere.

Similarly, the Euler characteristic of the torus is zero. This means that the torus requires at least seven vertices to triangulate. In particular a triangulation of the torus with seven vertices is possible and has a 3d representation called Csaszar's torus [2]. However, this triangulation is difficult to visualise and so although it is the easiest to compute with more frequently a different triangulation that involves nine vertices is used.

4.2 A general method for triangulating an orientable surface of genus g.

There is a theorem in topology called the Classification theorem that says that any non-empty, compact, connected 2-manifold is homeomorphic to one of the following: a 2-sphere, a connected sum of r tori where $r \geq 1$ or the connected sum of k projective planes $k \geq 1$. Connected here simply means that the topological space of interest cannot be written as the disjoint union of two or more non-empty open sets.

A surface homeomorphic to either a sphere or a connected sum of r tori is orientable and a surface homeomorphic to the connected sum of r projective planes is non-orientable. Therefore, any orientable surface is homeomorphic to either a sphere, or a connected sum of r tori.

The sphere can be triangulated with a tetrahedron. The torus however is slightly more difficult to triangulate and requires more thought. The torus is homeomorphic to a rectangle with both left and right sides associated and top and bottom associated as seen in the figure below:

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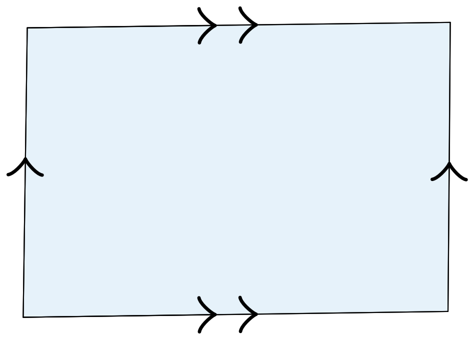

It is sufficient for us to triangulate the space instead because then our triangulation, K, will have associated polyhedron $|K|$ that is homeomorphic to this rectangle which is homeomorphic to a torus, and therefore the associated polyhedron will be homeomorphic to a torus and so a triangulation of a torus.

A triangulation of the torus appears below. This triangulation is frequently used in texts and in particular it is the one introduced in the text by Croom F.

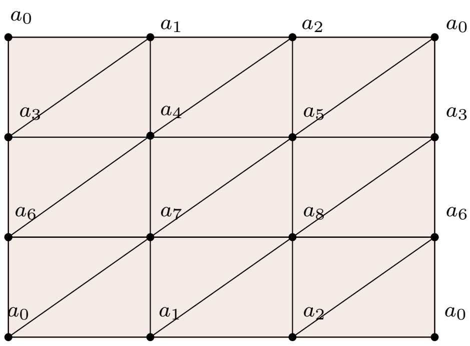

Figure 1 is a representation of this triangulation in three dimensional space.

This triangulation can be used as a building block for other triangulations of higher genus orientable surfaces, by 'glueing' appropriate edges together and removing several simplicies.

From the triangulation of the torus we can arrive at a triangulation for the two torus. The two torus is simply the connected sum of two 1-tori. This means that it is obtained by removing a disc from each of two 1-tori and connecting them with a cyclinder. An anaolgous procedure that can be performed on two polyhedra by simply removing one of the rectangular faces (this includes two 2-simplices and one

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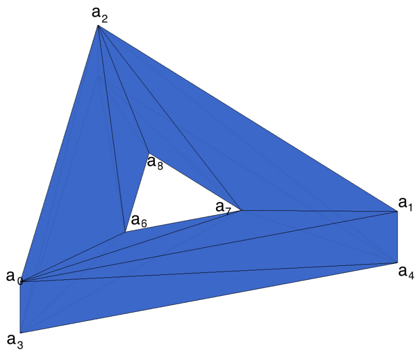

Figure 1: 3d Realisation of the Triangulation of a Torus

1-simplex) from each of the polyhedra, and pairing the corresponding sides of the rectangle to each other. A similar procedure can be repeated a number of times to produce a triangulation of an orientable surface of genus g, as illustrated for $g = 3$ in figure 2.

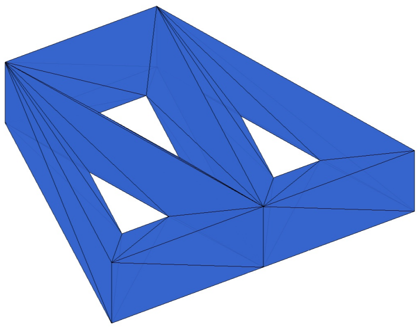

Figure 2: Triangulation of a Surface of Genus 3

Using this method of triangulating a surface of genus g I was able to prove the following theorem:

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Theorem 4.2. *If Σ is an orientable surface then the Euler characteristic of the surface $\chi(\Sigma) = 2 - 2g$ where g is a non-negative integer.*

5 The Gauss Bonnet Theorem

The Gauss Bonnet Theorem is a beautiful mathematical result that connects a surface's geometry to its topology. In particular, the Gauss Bonnet Theorem is as follows:

Theorem 5.1 (Gauss Bonnet Theorem- Continuous)**.** *If Σ is a closed oriented surface then:*

$$\int_\Sigma K dA = \chi(\Sigma) \tag{3}$$

where K is the Gaussian cature and χ is the Euler characteristic.

The Gaussian curvature at a point p of an orientable surface Σ is obtained as follows. First chose a direction for the normal vector which can always be done because the surface is orientable. Take a vector v tangent to Σ at p, then $\text{span}\{n, v\} \cap \Sigma$ is a curve in 3-dimensional space. Define the curvature, λ, at this point to be the reciprocal of the radius of the osculating circle, where here what is meant by osculating circle is a circle passing through p and two points on each side of p infinitesimally far apart.

The value of λ clearly depends on the tangent vector v chosen. The maximum and minimum curvatures λ_1 and λ_2 found by choosing v appropriately are called the principal curvatures at p. The product of these principle curvatures, $K = \lambda_1\lambda_2$, is the Gaussian curvature at p.

For a full proof of this result see Murray (2011).

The discrete version of this theorem (also called The Gauss Bonnet Theorem) is quite similar.

Theorem 5.2 (The Gauss Bonnet Theorem- Discrete)**.** *If K is a polyhedron with α_0 vertices, α_1 edges and α_2 faces then:*

$$\sum_{p \in K} \frac{K(p)}{2\pi} = \chi(K)$$

where K is zero at all points not vertices, and at the vertices it is defined to be the deficit angle at that vertex. Where:

$$\textit{Deficit at } p = 2\pi - \sum \textit{Angles around vertex.}$$

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Proof.

$$\begin{aligned}
\sum_{p\in K}\frac{K(p)}{2\pi} &= \sum_{p\in K}\frac{\text{Deficit at p}}{2\pi} \\
&= \sum_{p\in K}\frac{2\pi - \text{Angles at p}}{2\pi} \\
&= \alpha_0 - \frac{1}{2\pi}\sum_{p\in K}\text{Angles at p} \\
&= \alpha_0 - \frac{1}{2\pi}(\pi\alpha_2) \\
&= \alpha_0 - \frac{\alpha_2}{2} + \frac{3}{2}\alpha_2 - \frac{3}{2}\alpha_2 \\
&= \alpha_2 - \alpha_1 + \alpha_0 \\
&= \chi(K)
\end{aligned}$$

The proof currently only holds true for polyhedron with triangular faces. However, using a similar argument to that we used when proving the Eule-Poincare Theorem is a generalisation of Euler's theorem we would easily be able to extend this.

□

6 Conclusion

This report has been intended to provided the reader with an appreciation of what I have been working with for the past six weeks. I began with some preliminaries followed by a statement of the Euler-Poincare theorem which we showed was a generalisation of the Euler's theorem. I have ommited any lengthy proofs of the Euler-Poincare theorem as I do not believe they have a place here in my report, however I suggest the interested reader should investigate this further in Croom's text.

I then moved to a discussion of the research topics. The first was about the trade off between how easy a triangulation is to draw and how easy it was to compute with. The second regarded finding a general way to triangulate an orientable surface and we concluded with a mention of the discrete Gauss Bonnet theorem which I was able to prove.

This summer has been an invaluable experience for me. I have met new people, learnt mathematics that I wouldn't have encountered until honours otherwise and I have developed a better understanding of what honours might involve. I would like

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to thank my supervisor Professor Michael Murray as well as AMSI for making this experience possible. I would certainly recommend this to future students.

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[2] Lutz, F. H 2002, Csaszar's Torus, accessed 5/01/2013, http://www.eg-models.de/models/Classical_Models/2001.02.069/_direct_link.html

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