An introduction to the AJ Conjecture

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1 Abstract

This paper aims to give an introduction to the AJ conjecture, which relates two knot invariants. The coloured Jones function is a knot invariant derived from irreducible representations of $\mathfrak{sl}_2$. It has been shown by S. Garoufalidis that the coloured Jones function is q-holonomic, i.e. it satisfies a nontrivial recursion relation. The AJ conjecture proposes that a certain polynomial, which is determined by the coloured Jones function, is in fact the A-polynomial.

2 The A-Polynomial

2.1 Definition

The A-polynomial is an invariant for knots derived from the fundamental group of a given knot. The fundamental group and peripheral subgroup, as a complete knot invariant, are difficult to work with, so the A-polynomial is of interest as a way of simplifying the use of these. It is found by looking at representations of the fundamental group of a knot into $SL_2(\mathbb{C})$. There is no known combinatorial procedure for calculating the A-polynomial from a knot diagram.

Suppose a knot K has fundamental group G. If we have a loop that passes around only one strand of the knot, called a meridian, we can find the corresponding element of the fundamental group. Call this μ. Similarly, find a loop in space that travels parallel to the knot, such that the knot and loop have linking number zero. This is the longitude of the knot, called λ. Note that μ and λ will commute.

Now suppose we have a representation, ρ, of G into $SL_2(\mathbb{C})$. Any given representation will be conjugate to one that is upper triangular, so we will restrict to upper triangular representations. As matrices in $SL_2(\mathbb{C})$ have determinant 1, we can say that μ and λ are sent to the following matrices

$$\rho(\mu) = \begin{pmatrix} m & t \\ 0 & m^{-1} \end{pmatrix} \qquad \rho(\lambda) = \begin{pmatrix} l & k \\ 0 & l^{-1} \end{pmatrix},$$

where t, k, m and l are complex numbers. We will only use the top left entries of these matrices.

Consider the set of all such points (l, m), across all representations. Call this set S. S consists of several components of various dimensions. Take a component C, and consider its Zariski closure $\overline{C}$. $\overline{C}$ is the zero set of a set of polynomials. If $\overline{C}$ is the zero set of only a single polynomial, call this polynomial F_C. Define the product of all such polynomials F_C to be A'.

Finally, any polynomial found this way will by divisible by $(l-1)$, which comes from the abelian representations of G. Thus, we will divide A' out by $(l-1)$ to obtain the final A-polynomial $A_K(l, m)$.

2.2 Properties of the A-polynomial

Theorem 1. *If K is the unknot, $A_K(l, m) = \pm 1$.*

Proof. The fundamental group of the unknot is $\mathbb{Z}$, which is abelian. Hence there only exist abelian representations; upon dividing out by $(l-1)$ we are left with ± 1.

Theorem 2. *$A_K(l, m) = A_K(l^{-1}, m^{-1})$, up to multiplication by powers of l and m.*

Proof. We know that $\rho(\mu)$ and $\rho(\lambda)$ are commuting upper triangular matrices. We can therefore assume, conjugating if necessary, that

$$\rho(\mu) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \qquad \rho(\lambda) = \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix},$$

We can then conjugate by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ to obtain a new representation ρ' such that

$$\rho'(\mu) = \begin{pmatrix} m^{-1} & 0 \\ 0 & m \end{pmatrix} \qquad \rho'(\lambda) = \begin{pmatrix} l^{-1} & 0 \\ 0 & l \end{pmatrix},$$

Thus, for every point (l, m) in S, the point (l^{-1}, m^{-1}) is also in S. Hence, the zeroes of $A_K(l, m)$ are exactly the zeroes of $A_K(l^{-1}, m^{-1})$, and so these polynomials are identical up to multiplication by powers of l and m.

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2.3 A-polynomial of the figure eight knot

As an example, we will find the A-polynomial of the figure eight knot. To calculate the A-polynomial, we first find the Wirtinger presentation of the Figure eight knot's fundamental group. This is

$$< a, d | ada^{-1}dad^{-1}a^{-1}da^{-1}d^{-1} >,$$

where a and d are loops that pass around a component of the figure eight knot. By drawing a line parallel to the figure eight knot with linking number zero, we can determine a longitude of the knot. This is found to be $\lambda = d^{-1}ada^{-1}d^{-1}a^{-1}dad^{-1}a$. We take a meridian of the knot to be $\mu = a$. Let ρ be a representation of the group into $SL_2(\mathbb{C})$. We know from the Wirtinger presentation that $d = cac^{-1}$. Hence $\rho(d) = \rho(c)\rho(a)\rho(c^{-1})$, and so the representations of a and d are conjugate matrices. They can therefore be written in the form

$$\rho(a) = \begin{pmatrix} M & 1 \\ 0 & M \end{pmatrix}$$

$$\rho(d) = \begin{pmatrix} M & 0 \\ t & M \end{pmatrix}$$

We then use the group relation to find a new matrix, $R = \rho(ada^{-1}dad^{-1}a^{-1}da^{-1}d^{-1})$, whose entries are in terms of M and t. As this relation is the identity, we solve the equation $R = I$, where I is the identity matrix. This gives the solution

$$t = \frac{-1 + 3M^2 - M^4 - \sqrt{1 - 2M^2 - M^4 - 2M^6 + M^8}}{2M^2}$$

We also use the formula for the longitude to find $\rho(\lambda)$. This is quite large, but substituting in the above value of t gives a matrix with top left entry

$$L = -\frac{-1 + M^2 + M^6 - M^8 + \sqrt{1 - 2M^2 - M^4 - 2M^6 + M^8}}{2M^4}$$

$$+\frac{-M^4(-2 + \sqrt{1 - 2M^2 - M^4 - 2M^6 + M^8})}{2M^4}$$

This then gives the A-polynomial of the knot, as it satisfies the equation

$$-L + LM^2 + M^4 + 2LM^4 + L^2M^4 + LM^6 - LM^8 = 0$$

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3 q-holonomicity of the Coloured Jones Function

3.1 The coloured Jones function

The Jones polynomial is a knot invariant discovered by Vaughan Jones in 1984. More recently, a way of generalising this invariant was found, using irreducible representations of $\mathfrak{sl}_2$. The coloured Jones function

$$J_N(K) : \mathbb{N} \to \mathbb{Z}[q^{\pm 1/4}]$$

is a sequence of Laurent polynomials that measures the Jones polynomial of the cables of a knot. In this section, we will prove some useful properties of the coloured Jones function.

3.2 Definition of q-holonomicity

In general, only a small number of knots have a closed form formula for the coloured Jones function. If this can't be found, then it would be useful to know if the coloured Jones function satisfies a recursion relation. Discrete functions that satisfy some non-trivial recursion relation are called *q-holonomic*. For instance, the coloured Jones function of the trefoil knot satisfies the relation

$$J_n(K) = \frac{q^{n-1} + q^{4-4n} - q^{-n} - q^{1-2n}}{q^{1/2}(q^{n-1} - q^{2-n})} J_{n-1}(K) + \frac{q^{4-4n} - q^{3-2n}}{q^{2-n} - q^{n-1}} J_{n-2}(K).$$

A useful property of such functions is that, under certain operations, new q-holonomic functions can be assembled from known ones. We will make use of the fact that

Lemma 3. The set of q-holonomic functions is closed under the operations

- Sums and products of q-holonomic functions
- Multisums of q-holonomic functions, i.e. functions of the form

$$g(a, b, n_2, \ldots, n_m) = \sum_{n_1=a}^{b} f(n_1, n_2, \ldots, n_m)$$

This will be used to show that the coloured Jones function is q-holonomic, by assembling it from other q-holonomic functions.

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3.3 Definition of the coloured Jones function

The coloured Jones function is determined by looking at the trace of a certain homomorphism between vector spaces.

Let V be a vector space of dimension N. If we have an isomorphism R that sends $V \otimes V$ to itself, we can associate this isomorphism to the crossings of a braid diagram. To each positive crossing in the braid diagram, we will associate R. To each negative crossing, we associate R^{-1}.

We require that R obeys the same relation as is required of the generators of the braid groups

$$(R \otimes Id_V)(Id_V \otimes R)(R \otimes Id_V) = (Id_V \otimes R)(R \otimes Id_V)(Id_V \otimes R).$$

This equation is known as the Yang-Baxter equation.

We will give an exact formula for the R-matrix later; for now, we can define the coloured Jones polynomial. Let β be a braid with n strands, and let B_n be the braid group generated by $\sigma_1, \ldots, \sigma_n$. We define the operator $\tau(\beta)$,

$$\tau(\beta) : V^{\otimes n} - > V^{\otimes n},$$

which is determined by the properties that

$$\tau(\sigma_i^{\pm 1}) = Id_V^{\otimes i-1} \otimes R \otimes Id_V^{\otimes n-i-1},$$

$$\text{If } \beta = \beta' \beta'' \text{, then } \tau(\beta) = \tau(\beta')\tau(\beta'').$$

$\tau(\beta)$ is the operator that sends a braid to the corresponding homomorphism between vector spaces.

We can now define the *quantum trace* of $\tau(\beta)$. Let $e_i, i = 0, ..., N-1$ be the standard basis of V. Let K be the linear endomorphism of $V^{\otimes n}$ given by

$$K(e_{i_1} \otimes \cdots \otimes e_{i_n}) = q^{(n(N-1)-2i_1-\cdots-2i_n)/2} e_{i_1} \otimes \cdots \otimes e_{i_n}$$

Lemma 4. *K is q-holonomic.* This follows from Lemma 3.

Then the quantum trace of $\tau(\beta)$ is defined as the trace of the function

$$\tilde{\tau}(\beta) = \tau(\beta) \times K^{-1}.$$

Specifically, the quantum trace is given by

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$$tr_q(\beta) = \sum_{1\leq i\leq n} \sum_{0\leq a_i\leq N} \tilde{\tau}(\beta)^{a_1,\ldots,a_m}_{a_1,\ldots,a_m}$$

If V is set to be the N-dimensional $U_q(\mathfrak{sl}_2)$-module, this quantum trace is the coloured Jones polynomial, $J_N(L)$. It can be shown that it is a knot invariant [5]. $J_N(L)$ has the properties that

- If K is the unknot, $J_N(K) = 1$.
- $J_2(L)$ is the Jones polynomial $J(q^{-1})$, up to multiplication by ± 1.

3.4 Proof that the coloured Jones function is q-holonomic

First, we will give an explicit formula for the R-matrix. Consider the function

$$f_+(N,a,b,k) := (-1)^k q^{-((N-1-2a)(N-1-2b)+k(k-1))/4} \begin{bmatrix} b+k \\ k \end{bmatrix} \{N-1+k-a\}_k,$$

where we are using the q-holonomic functions

$$\{n\} = q^n - q^{-n}, \quad \{n\}_k = \prod_{i=1}^{k} \{n-i+1\}, \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{\{n\}_k}{\{k\}_k}.$$

The central point to be made about f_+ is that

Lemma 5. *f_+ is q-holonomic in all variables.*

This follows from Lemma 3.

The entries of the R-matrix can then be defined in terms of these functions [5], as follows

$$(R)^{c,d}_{a,b} := f_+(N,a,b,c-b)\delta_{c-b,a-d},$$

$$(R^{-1})^{c,d}_{a,b} := f_+(N,a,b,b-c)\delta_{c-b,a-d},$$

where $\delta_{i,j}$ is the Kronecker delta function. It follows from Lemma 3 that $\tau(\beta)$ is q-holonomic. Thus $J_N(L)$ is q-holonomic, as it is a sum over q-holonomic functions.

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4 The AJ Conjecture

4.1 The non-commutative A-polynomial

A useful way of looking at a q-holonomic function is by examining the operator algebra of the recursion relations it satisfies. To do this, take the operators E and Q, which act on discrete functions $f : \mathbb{N}- > \mathbb{Z}[q^{\pm}]$ defined by

$$(Qf)(n) = q^n f(n) \qquad (Ef)(n) = f(n+1).$$

These operators satisfy $EQ = qQE$, and the algebra $\mathcal{A}$ generated by polynomials in E and Q, modulo this relation, is the set of possible recursion relations that f could satisfy. We say that f is q-holonomic iff there exists $P \in \mathcal{A}$ such that $Pf = 0$. The set $I_f = \{P \in \mathcal{A} | Pf = 0\}$ is known as the *recursion ideal* of f.

Unfortunately, $\mathcal{A}$ is not a principal ideal domain, so not all recursion ideals are generated by a polynomial in E and Q. Such a polynomial would be the non-commutative A-polynomial of an ideal, and applying this to the recursion ideal of $J_N(L)$ would give the A-polynomial of a knot. We solve this by instead considering the *Ore algebra* $\mathcal{A}_{loc} = \mathbb{K}[E, \sigma]$ over the field $\mathbb{K} = \mathbb{Q}(q, Q)$, where σ is defined by

$$\sigma(f)(q, Q) = f(q, qQ).$$

Multiplication of monomials is given by $aE^k \cdot bE^l = a\sigma^k(b)E^{k+l}$.

We can then define the recursion ideal of a function f with respect to $\mathcal{A}_{loc}$ in the same way as before. f is q-holonomic with respect to $\mathcal{A}_{loc}$ iff it is q-holonomic with respect to $\mathcal{A}$.

Now $\mathcal{A}_{loc}$ is a principal ideal domain [2], so we can find a generator $A_q(I_f)$ of the recursion ideal I_f with the desired properties

- $A_q(I_f)$ has the smallest E-degree and also lies in $\mathcal{A}$.
- We can write $A_q(I_f) = \sum_k a_k E^k$, where $a_k \in \mathbb{Z}[q, Q]$ are coprime.

These properties uniquely determine $A_q(I_f)$ up to left multiplication by $\pm q^a Q^b$.

We define A_q polynomial of a knot K to be the A_q polynomial of the recursion ideal of $J_N(K)$. Note that this ideal will be non-zero, as $J_N(K)$ is q-holonomic.

We will identify the *meridian, longitude* pair (M, L) with (Q, E) in the following way

$$M^2 = Q \qquad L = E$$

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Conjecture 6. *The AJ Conjecture*

For any knot K, $A(K)(L,M) = \epsilon A_q(K)(L,M^2)$, where ϵ is the evaluation map that sets $q = 1$.

The A_q polynomial can be computed using the WZ Algorithm, developed by Wilf-Zeilberger. This allows the AJ Conjecture to be verified in certain simpler cases. The conjecture is true for the 3_1 and 4_1 knots [3], and in the case of torus knots [1].

5 Conclusion

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