INTERNATIONAL CENTRE
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Kirby calculus and handle body theory

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My project has mainly considered surgeries on 3- and 4-manifolds, beginning first with the handlebody description of 4-manifolds. A *k*-handle *h* on an *n*-manifold is defined as a thickened *k*-disc, $D^k \times D^{n-k}$, boundary $\partial h = S^{k-1} \times D^{n-k} \cup D^k \times S^{n-k-1}$. We take the first of these components and use it to attach *h* to the boundary of a 4-manifold *M* via an embedding $\varphi : S^{k-1} \times D^{n-k} \to \partial M$, and then $\psi = \varphi|_{S^{k-1} \times S^{n-k}}$ attaches the second component to ∂M. Thus the boundary of the surgery manifold is given by $\partial M \setminus \varphi\left(S^{k-1} \times D^{n-k}\right) \cup_\psi D^k \times S^{n-k-1}$.

One of the first results is to see that every compact *n*-manifold can be constructed from empty space, by considering differentiable functions $f : M \to \mathbf{R}$, and the dense subset of these with nondegenerate critical points (the Morse functions). The compactness assumption on *M* means that there are only finitely many such points, and the nondegeneracy that locally we can find co-ordinates so that *f* appears as $f(x) = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_n^2$. In this case, the value of *k* can be shown to be equivalent to adding a *k*-handle.

There is also a duality between *k*- and (*n* – *k*)-handles (removing an (*n* – *k*)-handle is very much akin to adding a *k*-handle). As such, we need only encode information on *k*-handles, $k \leq \left\lfloor \frac{n}{2} \right\rfloor$. Moreover, the addition of 0- and *n*-handles can be homotoped so that we only need add *k*-handles ($k \neq 0, n$) to the *n*-disc. Finally, in the case of 4-manifolds, it can be shown that information about 1-handles can be encoded in the addition of 2-handles, so we need only consider attaching 2-handles to a 4-disc to construct any compact 4-manifold.

Since 2-handles are attached by embedding $S^1 \times D^2$ in $\partial D^4 = S^3$, we consider gluings of the form $M = S^3 \setminus \bigcup_{i=1}^{m} (\mathrm{Int} N_i) \cup_h \bigcup_{i=1}^{m} N_i$, where $N_i \cong S^1 \times D^2$ (disjoint tubular neighbourhoods of embeddings of S^1, i.e. the tubular neighbourhoods of components in a link), and *h* is the union of homeomorphisms $h_i : \partial N_i \to \partial N_i$. These are called Dehn surgeries on 3-manifolds, and to understand how they behave requires knowledge of homeomorphisms of the torus and solid torus. The former group are isomorphic, up to ambient isotopy, to $GL_2(\mathbf{Z})$ by considering what each homeomorphism does to the homology class of meridians μ, and longitudes λ (the associated matrix having first column the homology co-ordinates of the image of μ, and second for the image of λ). Homeomorphisms of the solid torus are a subset of these – namely, those with only a meridian twist.

Another important piece of information is that these surgeries can be characterised entirely by how each h_i acts on μ_i. Supposing that the induced map on homology classes is h_i^*, then if $h_i^*(\mu_i) = a_i\lambda_i + b_i\mu_i$, we define the surgery coefficient of N_i to be $r_i = \frac{b_i}{a_i} \in \mathbf{Q}^*$ (extended rational numbers). So by taking any link *L* in S^3, and assigning to each component an extended rational number, we specify a Dehn surgery. Much of my work was concerned with demonstrating the following propositions:

1) Attaching 2-handles to D^4 is equivalent to Dehn surgeries iff $r_i \in \mathbf{Z}$ for all *i*.

2) Dehn surgery on a knot results in a homology sphere if and only if $r^{-1} \in \mathbf{Z}$.

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