FINITE SIMPLE GROUPS, TREES AND SHABAT POLYNOMIALS

WILLIAM CRAWFORD

This is a report on a summer research project conducted in the School of Mathematical Sciences at the University of Adelaide under the supervision of Associate Professor Finnur Lárusson from November 2011 to January 2012. The author was supported by an AMSI Vacation Research Scholarship.

Contents

1. Background

1.1. **Unramified Coverings.** An *unramified covering* of a topological space X is a topological space Y and a map $f : Y \to X$ such that:

(1) f is continuous and surjective.
(2) for any $x \in X$ there exists a neighbourhood V of x such that the inverse image $f^{-1}(V)$ is the disjoint union of subsets of Y, each of which is homeomorphic to V by f. In other words $f^{-1}(V)$ is homeomorphic to $V \times S$ where S is a discrete space.

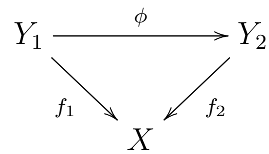
If we consider the case where X is path connected then it follows that $f^{-1}(V)$ consists of $|S|$ path connected components, called *sheets*.

1.2. **Ramified Coverings.** A topological space Y and a map $f : Y \to X$ is a *ramified covering* of a topological space X if there is a non-empty finite subset $R \subset X$ such that $Y \backslash f^{-1}(R)$, with the restriction $f|_{Y \backslash f^{-1}(R)}$ of f, is an unramified covering of $X \backslash R$. Nontriviality also requires that $|f^{-1}(x)| < |S|$ for all $x \in R$. The points of R are called the *ramification points* of the covering or the *critical values* of f, while the points in the inverse image $f^{-1}(R)$ are called the *critical points* of f.

Date: 22 January 2012.

As a simple example consider the map $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $f(z) = z^n$. The inverse image of a (sufficiently small) neighbourhood, V, of a point $z \in \mathbb{C} \setminus \{0\}$ is homeomorphic to n copies of V. Hence $\mathbb{C} \setminus \{0\}$ together with f is an unramified covering of $\mathbb{C} \setminus \{0\}$. If we add the point 0 to both spaces and define $\tilde{f} : \mathbb{C} \to \mathbb{C}$ by $\tilde{f}(z) = z^n$ then $\tilde{f}$ and $\mathbb{C}$ form an n-sheeted covering of $\mathbb{C}$ ramified at 0.

Lastly two coverings, $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$, are said to be *isomorphic* if there is an homeomorphism $\phi : Y_1 \to Y_2$ such that the following diagram is commutative:



1.3. **Homotopy and the Fundamental Group.** Let X be a topological space and take a point $x_0 \in X$. A *path* in X is a continuous map γ from the interval $[0, 1]$ to X. A *loop* in X is a path with the same start and end points, then $\gamma(0) = \gamma(1)$ is called the *base point* of the loop.
Two paths γ_1 and γ_2 are said to be *homotopic* if there is a continuous map $f : [0, 1]^2 \to X$ such that $f(s, 0)$ and $f(s, 1)$ are fixed for all s and $f(0, t) = \gamma_1(t)$ and $f(1, t) = \gamma_2(t)$ for all $t \in [0, 1]$. The map f is called an homotopy. An homotopy is often thought of as a continuous deformation of one path into another.
Now consider the set U of all loops with base point x_0. The *fundamental group* of X with base point x_0, denoted $\pi_1(X, x_0)$, is the set of equivalent classes of U under the equivalence relation of homotopy (that is two loops are said to be equivalent if they are homotopic). The product of two (equivalence classes of) loops $\gamma_1, \gamma_2 \in \pi_1(X, x_0)$ is defined as (equivalence class of) the loop that follows first γ_1 and then γ_2, that is

$$\gamma_1\gamma_2 = \begin{cases} \gamma_1(2t) & \text{if} \quad 0 \leq t \leq 1/2, \\ \gamma_2(2t-1) & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

This operation is clearly well defined (the product of two loops must still be a loop with the same base point) with identity the constant loop $\gamma = x_0$. The inverse of a loop γ is simply the loop going in the opposite direction $\gamma^{-1}(t) = \gamma(1-t)$ since their product is homotopic to the identity.

In the context of covering spaces there is a natural way in which the fundamental group $\pi_1(X, x_0)$ acts on the preimage of the base point $f^{-1}(x_0)$, called *monodromy.*

1.4. **Monodromy.** Let (Y, f) be an n-sheeted unramified covering of X, and let $E = f^{-1}(x_0)$ for some $x_0 \in X$, so $|E| = n$. For each loop $\gamma \in \pi_1(X, x_0)$ the inverse image $f^{-1}(\gamma)$ consists of n paths in Y all of which start and end at points in E, hence we can define $g : E \to E$ that maps the start point of each of these paths to the end point. The initial loop γ was invertible in $\pi_1(X, x_0)$, so there is a corresponding inverse of the map g. Hence g is a bijection from E to E.
Now consider $\phi : \pi_1(X, x_0) \to Aut(E)$ with $\gamma \mapsto g$, this is a group homomorphism since the target of the product $\gamma_1\gamma_2$ of two loops is a map g which equals the composition $g_2 \circ g_1$ where $\alpha(\gamma_1) = g_1$ and $\alpha(\gamma_2) = g_2$. So $\gamma * x = g(x)$, for $x \in E$, is a group action called the *monodromy action* of $\pi_1(X, x_0)$ on E.
The viewpoint that monodromy corresponds to a homomorphism from the fundamental group to the symmetric group on n elements will be useful later on, while the group action and the monodromy group itself will not be of much interest.

1.5. **Mathieu Groups.** First we need the concept of an *n-transitive* action of a group G on a set X.

Definition 1. *A permutation group G acts n-transitively on a finite set X if for any two ordered subsets of X of size n, $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$, there is $g \in G$ such that $gx_1 = y_1$, $gx_2 = y_2$, etc.*

While looking for groups that acted multiply transitively Mathieu discovered two permutation groups, M_{24} and M_{12}, that acted 5-transitively on the sets of 24 and 12 elements respectively. Additionally he showed that there were subgroups $M_{23} \subset M_{24}$ and $M_{11} \subset M_{12}$ that acted 4-transitively on the sets of 23 and 11 elements respectively, and a subgroup $M_{22} \subset M_{23}$ that acted 3-transitively on the set of 22 elements. These were later shown to be simple and are now recognised as the smallest sporadic simple groups. For all $n \geq k$, S_n and A_{n+2} act k-transitively on their respective sets. It turns out that apart from S_n and A_{n+2} for $n \geq 4$ the only groups that act 4-transitively are the Mathieu groups M_{23} and M_{11}, and the only groups that act 5-transitively are the Mathieu groups M_{24} and M_{12}. As such the Mathieu groups M_{24}, M_{23}, M_{12} and M_{11} can be characterised by this property and the set they act on, though a proof of this requires the classification of simple groups [4].

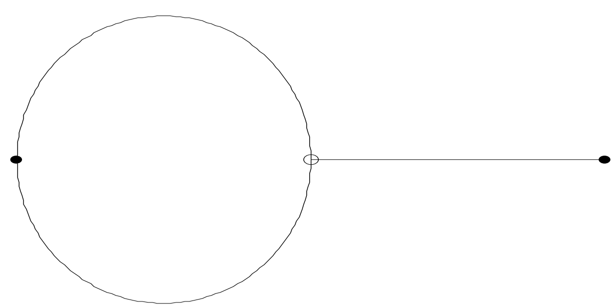
2. General Picture

2.1. **Dessins d'Enfants.** A *dessin d'enfant* is a finite connected graph with the following extra structure.

(1) the graph is bicoloured (to enable this the graph must be bipartite).
(2) edges connect to one vertex of each colour.
(3) at each vertex a cyclic ordering of the edges is given.

An example can be seen in figure 1

Figure 1. A simple example of a dessin d'enfant



2.2. **Permutations.** Two permutations acting transitively on a finite set E (that is, for any $x, y \in E$ there is a word g of the two permutations such that $gx = y$). Or equivalently a group homomorphism from the free group on two generators F_2 to the automorphism group of E, $\phi : F_2 \to Aut(E)$, such that the image acts transitively on E.

2.3. **Belyi Maps.** A meromorphic ramified covering $f : X \to \mathbb{P}$ with X a compact, connected Riemann surface, branched over at most three points which can be taken to be 0, 1 and ∞ (up to isomorphism).

Claim. *There is a naturally defined bijective correspondence between the three classes above.*

$$\textit{Dessin d'Enfants} \Longleftrightarrow \textit{Permutations} \Longleftrightarrow \textit{Belyi Maps}$$

There are a number of different directions to address here.

First **Permutations $\iff$ Dessin d'Enfant** is clear: Let the orbits of the two permutations be the black and white vertices respectively, then the elements of E are the edges. Transitivity implies the graph is connected, and the cyclic ordering is determined by the permutations. It's not hard to see that if we are given a dessin with n edges then labelling the edges (from 1 to n) will determine two permutations (so the correspondence is surjective). The cyclic ordering makes this correspondence injective (up to isomorphism of graphs).

Next **Belyi Maps $\iff$ Permutations**. If we start with a Belyi map $f : X \to \mathbb{P}$ then the critical values of f are 0, 1 or ∞, so f restricted to $X \setminus f^{-1}(\{0,1,\infty\})$ forms an unramified covering of the punctured sphere $\mathbb{P} \setminus \{0,1,\infty\}$. Then for a point $y_0 \in \mathbb{P} \setminus \{0,1,\infty\}$ the fundemental group $\pi_1(\mathbb{P} \setminus \{0,1,\infty\}, y_0)$ is the free group on two generators, namely $\pi_1(\mathbb{P} \setminus \{0,1,\infty\}, y_0) = \langle \gamma_1, \gamma_2 \rangle$ where γ_1 and γ_2 are the loops going once clockwise around 0 and 1 respectively. Since $\mathbb{P}$ is the sphere the loop going around ∞ is just the product $\gamma_1^{-1}\gamma_2^{-1}$. Then the monodromy action of $\pi_1(\mathbb{P} \setminus \{0,1,\infty\}, y_0)$ on $f^{-1}(y_0)$ determines a group homomorphism that maps γ_1 and γ_2 to two permutations $g_1, g_2 \in S_n$, where $n = |f^{-1}(y_0)|$. Since X is a connected Riemann surface it is path connected, hence for any $x, y \in f^{-1}(y_0)$ we can take a path from x to y in X and map it to a loop in $\mathbb{P} \setminus \{0,1,\infty\}$ that starts and ends at y_0. Therefore the permutations g_1, g_2 act transitively on $f^{-1}(y_0)$. Isomorphic covering spaces have isomorphic permutations (in the sense that there is a bijection $h : f^{-1}(y_0) \to \tilde{f}^{-1}(\tilde{y}_0)$ with $g_i = h^{-1} \circ \tilde{g}_i \circ h$) so the correspondence is injective. All that's left to show is surjectivity, which follows from Riemann's existence theorem as follows:

Given a pair of permutations α_1 and α_2 we can choose any three points $\{y_1, y_2, y_3\} \subset \mathbb{P}$ and apply Riemann's existence theorem to show that there is a compact Riemann surface X and a meromorphic function $f : X \to \mathbb{P}$ such that X and f form a covering of $\mathbb{P}$ with y_1, y_2 and y_3 the critical values of f and for a point $y_0 \in \mathbb{P} \setminus \{y_1, y_2, y_3\}$ the monodromy homomorphism maps the loops about y_1 and y_2 to α_1 and α_2 respectively. Furthermore for a given set of points $\{y_1, y_2, y_3\}$, f is unique up to isomorphism. Since the permutations α_1 and α_2 act transitively on $f^{-1}(y_0)$ for any $y_0 \in \mathbb{P} \setminus \{y_1, y_2, y_3\}$ the Riemann surface X must be path connected. If we specify that the critical values are 0, 1 and ∞ then the resulting covering of $\mathbb{P}$ is a Belyi map. Hence there is a bijective correspondence between Belyi maps and pairs of permutations that act transitively.

The actual construction of the Belyi map is as follows: Let α_1, α_2 be two permutations acting transitively on a finite set E. Take some $y_0 \in \mathbb{P} \setminus \{0,1,\infty\}$ and let $\phi : \pi_1(\mathbb{P} \setminus \{0,1,\infty\}, y_0) \to G = \langle \alpha_1, \alpha_2 \rangle$ be the group homomorphism taking $\gamma_1 \mapsto \alpha_1$ and $\gamma_2 \mapsto \alpha_2$ where γ_1 and γ_2 are as above. For a point $x \in E$ the stabiliser G_x of x in G is a subgroup of G, so the inverse image $M = \phi^{-1}(G_x)$ is a subgroup of $\pi_1(\mathbb{P} \setminus \{0,1,\infty\}, y_0)$. Now consider the set of paths in $\mathbb{P} \setminus \{0,1,\infty\}$ that start at y_0. Define an equivalence relation on this set by saying two paths α, β are equivalent if they end at the same point and the loop $\alpha\beta^{-1}$ is in M. Now take the set of equivalence classes as the space X and let f map each equivalence class to the common end point of the paths in it. The complex structure on X is then determined by lifting it from $\mathbb{P} \setminus \{0,1,\infty\}$ in such a way that f restricted to $X \setminus f^{-1}(\{0,1,\infty\})$ is holomorphic (I didn't look into the exact procedure). With this structure X is a compact Riemann surface which must be connected as described above, and f is a meromorphic function from X to $\mathbb{P}$.

Finally if we begin with a dessin d'enfant, then there is a corresponding pair of permutations, α_1 and α_2, and in turn a corresponding Belyi map. To get a dessin d'enfant from a Belyi map we may attempt to find the monodromy permutations corresponding to the generators of the fundamental group and then draw the dessin from the permutations, or we can shortcut this by taking the inverse image $f^{-1}([0,1])$ of the line segment $[0,1]$. Here we colour the points in the fibre of 0 black and the points in the fibre of 1 white, the preimages of the line segment (0,1) are the edges of the graph. The result is a dessin d'enfant embedded in the Riemann surface X (that the graph is connected is easiest to see by noting that the monodromy permutations act transitively).

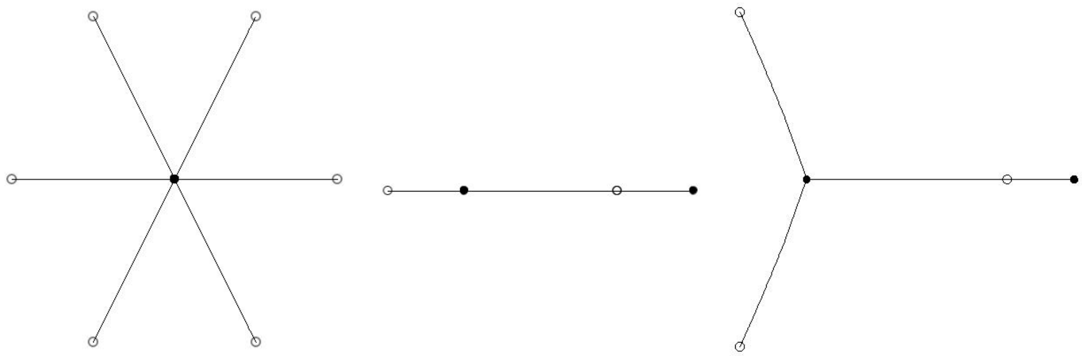
3. Restricted Picture

3.1. **Trees and Shabat Polynomials.** Suppose the dessin is a *planar tree*, that is, it contains no loops or circuits or equivalently it has $n+1$ vertices, where n is the number of edges. Then consider the corresponding Belyi map $f : X \to \mathbb{P}$. As described above $f^{-1}([0,1])$ is an embedding of the tree in X, since the tree is planar X must be the sphere $\mathbb{P}$. Hence f is a meromorphic function from $\mathbb{P}$ to $\mathbb{P}$, that is, a rational function. Further the only pole of f is ∞ (that is, $f(x){=}\infty \iff x{=}\infty$) so f is just a polynomial (otherwise the roots of the denominator would be poles of f). Hence we can ignore the mapping $f(\infty) = \infty$ and treat f as a complex valued polynomial $\mathbb{C} \to \mathbb{C}$ with at most two critical values, which may be taken to be 0 and 1. Such a polynomial is called a *Shabat polynomial*. Two Shabat polynomials P and Q are said to be equivalent if there are automorphisms ϕ and ψ of $\mathbb{C}$ such that $\psi \circ P = Q \circ \phi$. The question then arises, is the dessin d'enfant corresponding to every Shabat polynomial a tree?

Take a Shabat polynomial $P : \mathbb{C} \to \mathbb{C}$. We may assume it has any critical values at 0 or 1. Consider the preimage $P^{-1}([0,1])$. We want to show that this is a tree embedded in the complex plane. Firstly, as before let the inverse images of 0 be white vertices, the inverse images of 1 be black vertices and the inverse images of the line segment (0,1) be edges, then clearly we have a planer bicoloured graph. The inverse images of (0,1) must start at a point of $P^{-1}(0)$ and end at a point of $P^{-1}(1)$, hence edges connect to one vertex of each colour. Now suppose the graph contains a circuit, then the circuit bounds a region of the complex plane R, on the boundary of R, P takes only real values (in [0,1]). That is, the imaginary part of P is zero on the boundary of R, but since it is an harmonic function it must be zero on all of R (by the maximum principle), and hence on all of $\mathbb{C}$. But then the Cauchy-Riemann equations imply P is constant, contradicting $P^{-1}([0,1])$ having a circuit. So $P^{-1}([0,1])$ is a planar, bicoloured graph with no circuits, all that is left is to show that it is connected. If P has degree n then the number of solutions to $P(z) = 0$ and $P(z) = 1$ is $2n$ *counting multiplicities*. The number of solutions with multiplicity greater than 1 is the number of solutions of $P'(x) = 0$ which is $n - 1$. Hence the number of *distinct* solutions is $2n - (n - 1) = n + 1$, but each distinct solution is a vertex of $P^{-1}([0,1])$ which has n edges. Therefore $P^{-1}([0,1])$ is a tree.
The correspondence between trees and Shabat polynomials is thus surjective, and injective if we take into account equivalence classes of Shabat polynomials and isomorphism classes of graphs.

3.2. **The Corresponding Permutations.** The last part of the picture concerns the corresponding permutations, what condition must they satisfy if the dessin is a tree? We need some restriction on the permutations so that the dessin doesn't have any loops. One way of ensuring this is to specify that the group G generated by the permutations acts freely on the set $E = \{0, 1, \ldots, n\}$, that is, if $x \in E$ and $g \in G$ then $g * x = x$ iff g is the identity. A loop can only occur if there is a non-identity element that fixes a point, so specifying that G acts freely ensures that the result is a tree. However this is too strong. The following example shows that a group can have two corresponding dessins d'enfants only one of which is a tree, demonstrating that we must search for a property of the permutations not the group generated by them.

FIGURE 2. Planar Trees



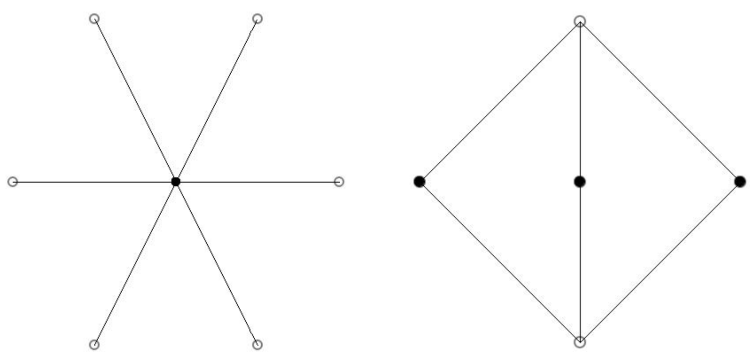
(A) A star

(B) A chain

(C) The simplest planar tree that is not a star or chain

Example. Take the subgroup isomorphic to $\mathbb{Z}_6$ in S_6, namely $G = \langle e, (1\ 2\ 3\ 4\ 5\ 6)\rangle$. Then the dessin corresponding to this pair of permutations is a 6-pointed star (and hence a tree). However we can also generate G as follows $G = \langle (1\ 3\ 5)(2\ 4\ 6),\ (1\ 4)(2\ 5)(3\ 6)\rangle$, then the corresponding dessin is not a star, in fact it's not even a tree! See figure 3.

FIGURE 3. Two possible trees corresponding to $\mathbb{Z}_6$



In fact the requirement on the permutations is that the total number of orbits of the two permutations is $n + 1$. Interestingly this implies that if a group acts freely on $\{1, 2, ..., n\}$ and can be generated by two permutations, then the number of orbits between the two permutations must be $n + 1$.

Clearly trees correspond bijectively with pairs of permutations that have a total of $n+1$ orbits (and that act transitively). We have shown that trees correspond to Shabat polynomials, so the picture is complete.

4. The Project

The applicability of the above picture is explained by a couple of results. First Cayley's theorem states that every group is isomorphic to a subgroup of S_n for some n, this is called the *permutation representation* of the group (note that a permutation representation is by no means unique). Clearly a permutation representation of a finite group is then generated by a finite set of permutations (sometimes themselves referred to as the representation).
The natural construction of a permutation representation of a group G is the image of the

injective group homomorphism $\phi : G \to S_G$ defined by $\phi(g) = \alpha_g$ where α_g is the bijection from G to G defined by $\alpha_g(h) = gh$. That ϕ is an homomorphism and is injective (and also that α_g is a bijection) follows straight from the group axioms. Then we notice that the action of $im(\phi) = \{\alpha_g : g \in G\}$ on G is transitive, namely for any $x, y \in G$ we have $g = yx^{-1} \in G$ and hence $\alpha_g * x = gx = yx^{-1}x = y$.
The question then becomes: when can the naturally defined permutation representation of a group be generated by only two permutations? This is answered in the case of finite simple groups:

Theorem. *For any nontrivial element $g \in G$ of a finite simple group G there is another element h of G such that the pair generate G, that is $G = \langle g, h\rangle$.*

This was shown in [5] (The socle of a group G is the group generated by all minimal normal subgroups of G, in the case of a simple group this is just itself). Hence we know that any finite simple group can be represented by a pair of permutations that have a transitive natural action. For any such group there is then a corresponding dessin d'enfant and Belyi map. The goal of the project was to find a permutation representation for one of the sporadic simple groups with a tree as its corresponding dessin d'enfant, and then to attempt to calculate the corresponding Shabat polynomial.

4.1. **The trees corresponding to M_{11} and M_{23}.** I began by looking for permutation representations for the Mathieu groups on the group ATLAS [6]. The goal was to find representations of the Mathieu groups as subset of S_n for fairly small n, then it would be possible to draw the dessin. The Mathieu groups are defined on S_n where $n = 11$, 12, 22, 23 or 24, and for each Mathieu group the ATLAS gives a possible pair of permutations in S_n that generate M_n. Due to the characterisation of the Mathieu groups above one way of finding such permutations would be a random search, e.g. if we wanted to find permutations in S_{24} that generated M_{24} we could randomly search for a pair of permutations that generate a group of order $|M_{24}| = 10200960$ and that act 5-transitively on $\{1, 2, \dots, n\}$. Similarly representations for M_{11}, M_{12} and M_{23} could be found. M_{22} is the stabiliser of a pair of points in $\{1, 2, \dots n\}$ under the action of M_{24}, so we can use this to find a representation for it.

In the paper [7] it is shown that it is not possible to choose generators for M_{12}, M_{22} and M_{24} that result in the dessin being a tree. The authors also show that there is precisely one tree corresponding to M_{11} and two trees corresponding to M_{23}.
I wrote a program in Matlab that takes a pair of permutations, one of which must be an involution, and draws the corresponding dessin if it is a tree. Figures 4 and 5 show the trees corresponding to the following permutation representations for M_{11} and M_{23}:

$$M_{11} = \langle(2\ 10)(4\ 11)(5\ 7)(8\ 9),\ (1\ 4\ 3\ 8)(2\ 5\ 6\ 9)\rangle$$

$$\begin{aligned} M_{23,\,A} = \langle &(1\ 15)(2\ 13)(4\ 20)(5\ 6)(7\ 19)(9\ 11)(12\ 23)(16\ 18), (1\ 3\ 14\ 2) \\ &(4\ 8\ 19\ 18)(5\ 17)(6\ 16)(9\ 22\ 10\ 21)(11\ 20\ 15\ 12)\rangle \\ M_{23,\,B} = \langle &(1\ 2)(3\ 4)(7\ 8)(9\ 10)(13\ 14)(15\ 16)(19\ 20)(21\ 22), (1\ 16\ 11\ 3) \\ &(2\ 9\ 21\ 12)(4\ 5\ 8\ 23)(6\ 22\ 14\ 18)(13\ 20)(15\ 17)\rangle \end{aligned}$$

4.2. **Computing the Shabat Polynomial.** The Mathieu groups are the smallest sporadic simple groups, but further they also have permutation representations defined on the smallest number of points of all the sporadic simple groups. The computational complexity of calculating the Shabat polynomial for a tree increases very quickly as the number of vertices goes up, hence I attempted to calculate the Shabat polynomial for the smallest Mathieu group, M_{11}, and its corresponding tree.

Figure 4. The tree representing M_{11}

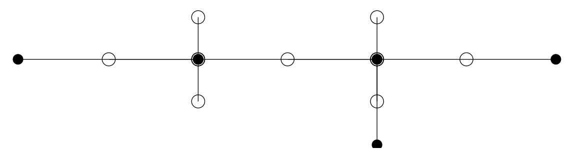
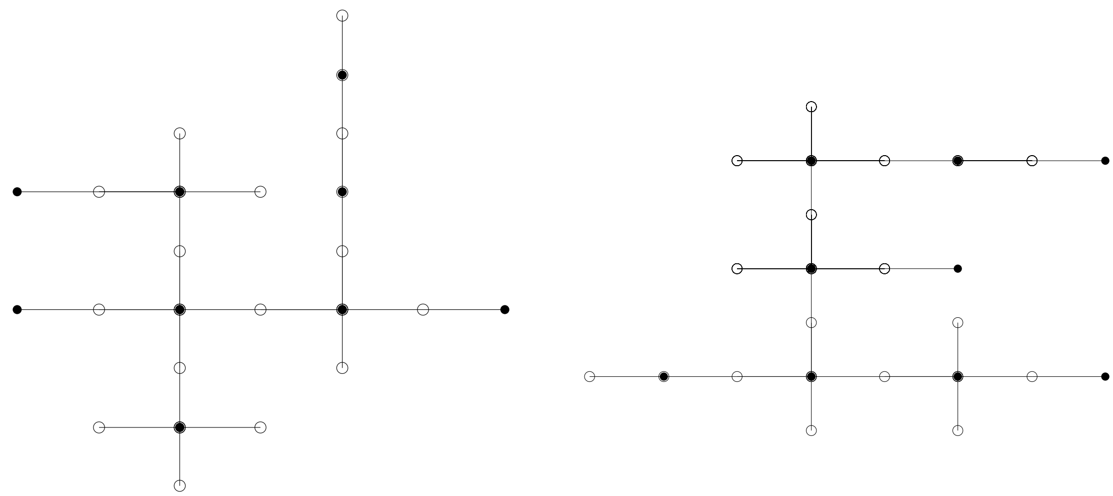


Figure 5. The two trees representing M_{23}



(A) (B)

The simplest cases for a tree are stars or chains, see figure 2. In the case of a star we have a Shabat polynomial with a single critical value, if we set this to be 0 then the fibre of 1 has n points in it so 1 is not a critical value of P. The Shabat polynomial must then be of the form $P(z) = z^n + C$, the choice of C determines where the tree is position on the complex plane. Specifying that the tree be centred at zero, i.e., $P^{-1}(0) = 0$, we get $C = 0$.

When the tree is a chain the corresponding Shabat polynomial is a Chebyshev polynomial $T_n(z) = \cos(n \arccos(x))$. This can be deduced since the Chebyshev polynomial $T_n(z)$ has only two critical values, ± 1, and hence is a Shabat polynomial. The corresponding tree has $n+1$ edges but $T_n(z)$ has $n-1$ critical points of degree 2 and hence $n-1$ vertices of valency 2. It follows that the tree must be a chain.

For other trees we have to put a bit more work in. Let a_1, a_2,..., a_p and b_1, b_2, ..., b_q be the positions of the black and white vertices respectively. Let α_1, α_2,..., α_p and β_1, β_2,..., β_q be the valencies of the vertices. For a given tree we know the number of black and white vertices and their valencies. If we specify that the critical values of the polynomial are 0 and 1 then it must satisfy the following equations:

$$P(z) = C(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_p)^{\alpha_p},$$
$$P(z) - 1 = C(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_q)^{\beta_q}.$$

That the dessin is a tree means $p+q = n+1$ where $n = \sum \alpha_i = \sum \beta_i$. Ensuring equality of the coefficients gives n equations in $n+2$ unknowns: C, a_1, a_2, ..., a_p, b_1, b_2, ..., b_q. We then have a couple of degrees of freedom corresponding to the choice of a Shabat polynomial in an equivalence class. In general I specified that $C = 1$ and also chose the position of one of the vertices. There is also the choice of specifying only one of the critical values and choosing the position of two of the vertices (as well as setting $C = 1$) instead, to reduce computational complexity it is generally easier to do this since we get the following equations:

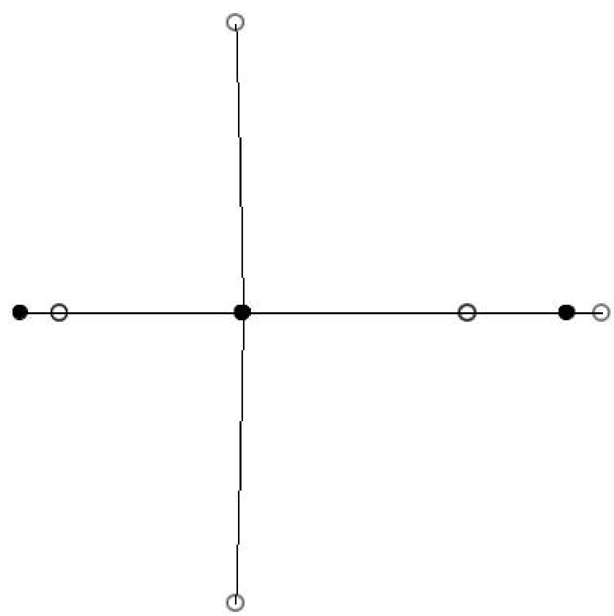
$$P(z) = C(z - a_1)^{\alpha_1}(z - a_2)^{\alpha_2} \cdots (z - a_p)^{\alpha_p},$$
$$P(z) - y_2 = C(z - b_1)^{\beta_1}(z - b_2)^{\beta_2} \cdots (z - b_q)^{\beta_q}.$$

But then there is only one equation involving the second critical value y_2 so we can eliminate it from the system of equations and solve for it later. The result is having to solve a system of $n - 1$ equations instead of n (once the vertices have been chosen).

Example. A simple example of this is finding a Shabat polynomial that corresponds to the simplest tree that is not a chain or a star, seen in figure 2. First let the critical value corresponding to the black vertices be 0. Then $P(z) = (z - a_1)^3(z - a_2)$, so we can choose $a_1 = 0$ and $a_2 = 1$, giving $P(z) = z^3(z - 1)$ and positioning the tree along the real axis. To find the second critical value, compute $P'(z) = z^2(4z - 3)$, which is zero at 0 and $3/4$. It follows that $P(z) - y_2 = (z - 3/4)^2(z - b_2)(z - b_3)$. So the white vertex of valency 2 is at $z = 3/4$ and $y_2 = P(3/4) = -27/256$.
Finally note that the positions of the last two white vertices b_1 and b_2 are roots of $z^3(z-1) = -27/256$.

For more complicated trees a computer algebra system must be used to solve the system of equations. As an example consider the problem of finding a Shabat polynomial corresponding to the tree in figure 6.

FIGURE 6. A tree with valencies 4, 2, 1 and 2, 2, 1, 1 ,1



Our system then looks like

$$P(z) = z^4(z - 1)^2(z - a_3),$$
$$P(z) - y_2 = (z - b_1)^2(z - b_2)^2(z - b_3)(z - b_4)(z - b_5).$$

Notice that I have already chosen the position of two of the black vertices. Before rushing into equating the coefficients to solve for a_3, b_1, b_2, b_3, b_4, b_5 and y_2, it is worth noting that in general the coefficients of the polynomials will come from a smaller number field than the one containing a_3, b_1, b_2, b_3, b_4 and b_5 as they are currently defined. Hence it is more computationally efficient to expand the brackets as follows:

$$P(z) = z^4(z - 1)^2(z - a_3),$$

$$P(z) - y_2 = (z^2 - b_1 z - b_2)^2 (z^3 - b_3 z^2 - b_4 z - b_5).$$

It is then easy to show (see [1], page 86) that the newly defined a_3, b_1, b_2, b_3, b_4 and b_5 come from the same number field as the coefficients of P. Now let's have a look at the equations between the coefficients:

$$b_2^2 b_4 + 2b_2 b_1 b_5 = 0,$$
$$b_2^2 b_3 + 2b_2 b_1 b_4 + (-2b_2 + b_1^2) b_5 = 0,$$
$$-b_2^2 + 2b_2 b_1 b_3 - 2b_1 b_5 + (-2b_2 + b_1^2) b_4 = 0,$$
$$-a_1 - 2b_2 b_1 + (-2b_2 + b_1^2) b_3 - 2b_1 b_4 + b_5 = 0,$$
$$1 + 2a_1 + 2b_2 - b_1^2 - 2b_1 b_3 + b_4 = 0,$$
$$-2 - a_1 + 2b_1 + b_3 = 0,$$
$$b_2^2 b_5 + y_2 = 0.$$

We have six equations in six unknowns, since we can eliminate the last equation and use it to calculate y_2 once the rest has been found. It is now possible to get a computer to solve this system and the resulting polynomial P will be a Shabat polynomial with the original tree as the inverse image of $[0, 1]$. The issue is that we have not specified any structure of the tree except the valencies of its vertices, so the system constructed is just as valid for every other tree that has black vertices with valencies 4, 2 and 1 and white vertices with valencies 2, 2, 1, 1 and 1. In fact solving the system of equations above gives us a number of solutions, each one corresponding to a possible tree with the specified valencies, see figure 7. Among these trees will be the one we started with and the Shabat polynomial corresponding to it is the one we set out to compute.
Here we see that the first tree is the one we started with and the corresponding Shabat polynomial is $P(z) = z^4(z-1)^2(z - \frac{1}{3} + \frac{2}{9}\sqrt{21})$.

$\mathbf{M}_{11}$: The only tree representing M_{11} has 5 black vertices with valencies 4, 4, 1, 1 and 1 and 7 white vertices with valencies 2, 2, 2, 2 1, 1 and 1. The Shabat polynomial then has the form:

$$P(z) = (z^2 + 2z - b_2)^4 (z^3 - b_3 z^2 - b_4 z - b_5),$$
$$P(z) - 1 = (z^4 - a_1 z^3 - a_2 z^2 - a_3 z - a_4)^2 (z^3 - a_5 z^2 - a_6 z - a_7).$$

Notice that I have specified the second critical value of the polynomial instead of choosing the position of 2 vertices. This is because none of the vertices are unique in the sense that they are the only vertex of a given colour and valency. Specifying 2 vertices results in every tree appearing multiple times, namely once for every possible permutation of the coordinates. As an example if I were to further specify that $b_2 = 0$ then the tree will have a black vertex of valency 4 at 0 and -2, however the direction of the tree is not specified! So unless the tree is mirror symmetric 2 solutions will be returned for every tree. This is a serious problem; doing twice as many computations as necessary. One way of avoiding this is to choose the values of two unknowns in such a way that the vertices are not fully specified. Alternatively I have chosen to specify the second critical value as 1. Equating coefficients we get the following grand set of equations to solve:

$$-b_2^4 b_4 + 8b_2^3 b_5 + a_4^2 a_6 + 2a_4 a_3 a_7 = 0,$$
$$-b_2^4 b_3 + 8b_2^3 b_4 - (2b_2^2(-2b_2 + 4) + 16b_2^2) b_5 + a_4^2 a_5 + 2a_4 a_3 a_6 + (2a_4 a_2 + a_3^2) a_7 = 0,$$
$$b_2^4 + 8b_2^3 b_3 - (2b_2^2(-2b_2 + 4) + 16b_2^2) b_4 - (8b_2^2 - 8b_2(-2b_2 + 4)) b_5 - a_4^2 +$$
$$2a_4 a_3 a_5 + (2a_4 a_2 + a_3^2) a_6 + (2a_4 a_1 + 2a_3 a_2) a_7 = 0,$$
$$-8b_2^3 - (2b_2^2(-2b_2 + 4) + 16b_2^2) b_3 - (8b_2^2 - 8b_2(-2b_2 + 4)) b_4 - (2b_2^2 - 32b_2 +$$
$$(-2b_2 + 4)^2) b_5 - 2a_4 a_3 + (2a_4 a_2 + a_3^2) a_5 + (2a_4 a_1 + 2a_3 a_2) a_6 + (-2a_4 + 2a_3 a_1 + a_2^2) a_7 = 0,$$
$$2b_2^2(-2b_2 + 4) + 16b_2^2 - (8b_2^2 - 8b_2(-2b_2 + 4)) b_3 - (-24b_2 + 32) b_5 - (2b_2^2 - 32b_2 + (-2b_2 + 4)^2) b_4 -$$
$$2a_4 a_2 - a_3^2 + (2a_4 a_1 + 2a_3 a_2) a_5 + (-2a_3 + 2a_2 a_1) a_7 + (-2a_4 + 2a_3 a_1 + a_2^2) a_6 = 0,$$

$$8b_2^2 - 8b_2(-2b_2+4) - (2b_2^2 - 32b_2 + (-2b_2+4)^2)b_3 - (-24b_2+32)b_4 - (-4b_2+24)b_5 -$$
$$2a_4a_1 - 2a_3a_2 + (-2a_4 + 2a_3a_1 + a_2^2)a_5 + (-2a_3 + 2a_2a_1)a_6 + (-2a_2 + a_1^2)a_7 = 0,$$
$$2b_2^2 - 32b_2 + (-2b_2+4)^2 - (-24b_2+32)b_3 - (-4b_2+24)b_4 - 8b_5 + 2a_4 -$$
$$2a_3a_1 - a_2^2 + (-2a_3 + 2a_2a_1)a_5 + (-2a_2 + a_1^2)a_6 - 2a_1a_7 = 0,$$
$$-24b_2 + 32 - (-4b_2+24)b_3 - 8b_4 - b_5 + 2a_3 - 2a_2a_1 + (-2a_2 + a_1^2)a_5 - 2a_1a_6 + a_7 = 0,$$
$$-4b_2 + 24 - 8b_3 - b_4 + 2a_2 - a_1^2 - 2a_1a_5 + a_6 = 0,$$
$$8 - b_3 + 2a_1 + a_5 = 0,$$
$$-b_2^4b_5 + a_4^2a_7 + 1 = 0.$$

It goes without saying that solving these algebraically is not easy. I used Maple to solve these equations, substitute all the possible solutions into P and plot the corresponding trees, which can be seen in figure 8. The first of these trees is the true geometric form of the tree representing M_{11} (one might even say the true geometric form of M_{11} itself!), unique up to rotation, scaling and translation. Labelling the edges of this tree from 1 to 11 fully represents all 7920 elements of M_{11} as a subgroup of S_{11}; namely the elements of M_{11} are permutations of $\{1, 2, \ldots, n\}$ that can be obtained by some combination of the black and white orbits. Unfortunately the corresponding Shabat polynomial, which also contains all information about M_{11}, takes pages to write out algebraically so I can't include it here (primarily because manually converting it from the output in Maple to a format that is presentable would take me a very long time). Here is a numerical approximation to the polynomial to 50 digits:

$$P(z) = (z^2 + 2z + 0.62581981964539345482174248761620331193146048869277-$$
$$0.65734856252381653452706737960207902664329826542180i)^4(z^3+$$
$$(3.7566015577331081325569139475218453529478339052774+$$
$$1.0944313205965140030605813539274592427367192481771i)z^2+$$
$$(3.4661772407380389794243482036126100426153170251484+$$
$$0.47469090763533357882548552019808243414880529292628 9i)z+$$
$$2.2008151423298537394243946493561217959684294515343-$$
$$3.2526336518287131019635867182279039131600930214990i)$$

And expanded to 4 digit accuracy:

$$P(z) = z^{11} + (11.76 + 1.094i)z^{10} + (60.04 + 6.598i)z^9 + (179.4 + 3.890i)z^8 + (350.5 - 66.76i)z^7$$
$$+(451.4-252.2i)z^6+(341.2-446.5i)z^5+(74.64-450.6i)z^4-(105.1+248.1i)z^3-(94.16+53.41i)z^2$$
$$+(-25.05 + 6.041i)z - 1.270 + 2.343i$$

To get an idea of the size of the thing here is the coefficient of z^{10}, which is by far the smallest of the coefficients:

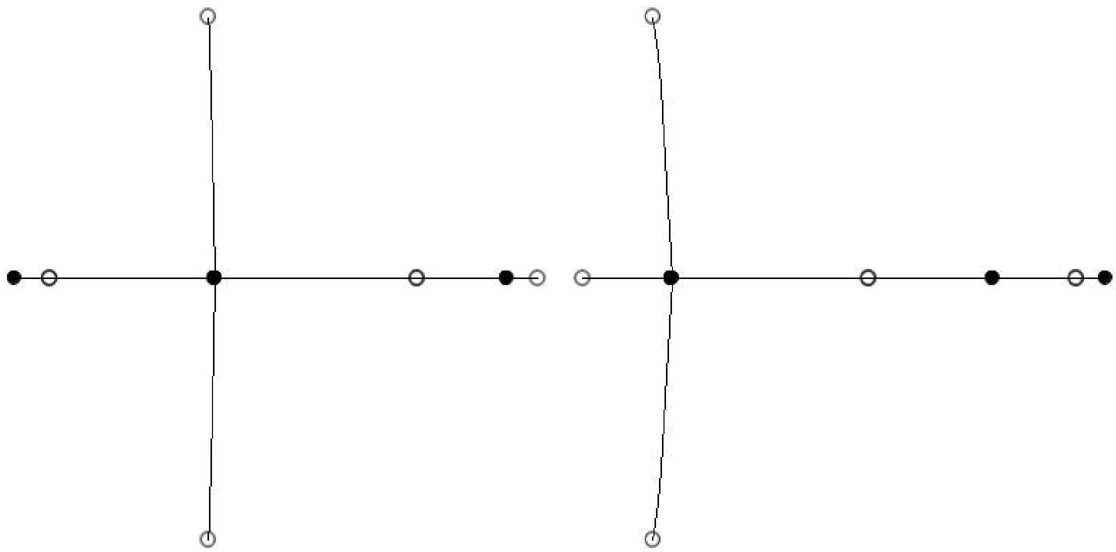
$$c_{10} = -\frac{33}{5} + \frac{11}{10}q$$

Where q is a root of the following polynomial:

$3993995989240812834048z^{20}+26626639928272085560320 0z^{19}+761028816023984015811072 0z^{18}+$
$119309324422726723444346880z^{17} + 1064681305509048069391053120z^{16} +$
$4566996167014004833513692288z^{15} - 3528511915314477194334676368z^{14} -$
$12806470045372685048135219404 8z^{13} - 40654226422018833988932226612 8z^{12} +$
$61995661049512985902655783083 2z^{11} + 44794948405262948162643715570 80z^{10} -$
$26210414325969176791205762486 64z^9 - 21615996344968609951605524118492z^8 +$
$4797110451541280308080644265222 0z^7 + 182624310535255316039303397993 07z^6 -$
$36510872764018189056310868103111 8z^5 + 52778304007699965726519749077769 7z^4 +$

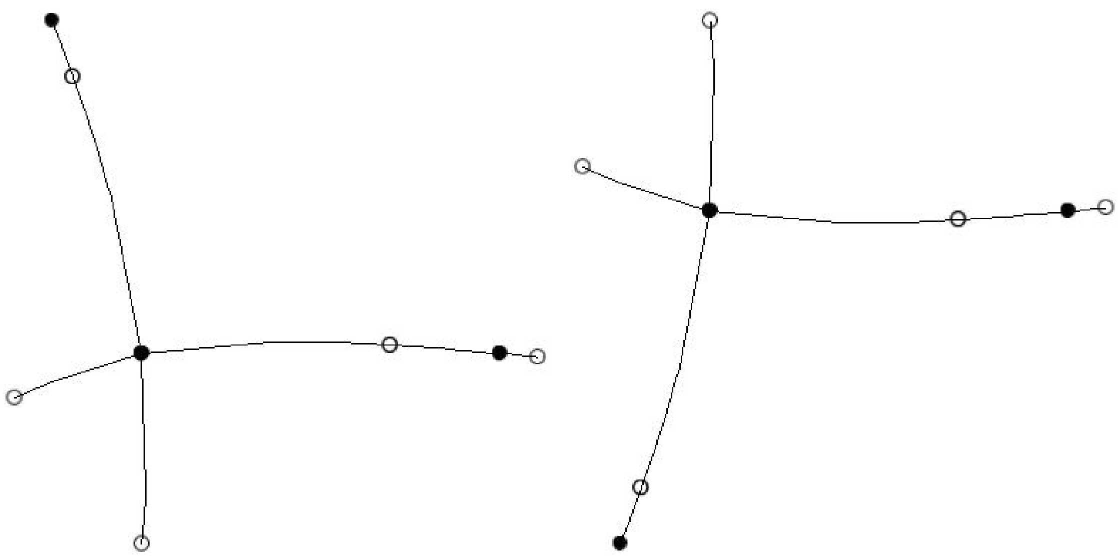
$4092627446969813296464821303496 60z^3 - 125415765149467585856085030044 4507z^2 +$
$7021090580116465987570342649500 56z - 156744756665424940479037029843408$

Figure 7. All trees with valencies 4, 2, 1 and 2, 2, 1, 1, 1 and corresponding Shabat polynomials



(a) $P(z) = z^4(z-1)^2(z - \frac{1}{3} + \frac{2}{9}\sqrt{21})$

(b) $P(z) = z^4(z-1)^2(z - \frac{1}{3} - \frac{2}{9}\sqrt{21})$



(c) $P(z) = z^4(z-1)^2(z + \frac{1}{4} - \frac{1}{4}\sqrt{-7})$

(d) $P(z) = z^4(z-1)^2(z + \frac{1}{4} + \frac{1}{4}\sqrt{-7})$

FIGURE 8. Trees with valencies 4, 4, 1, 1, 1 and 2, 2, 2, 2, 1, 1 ,1

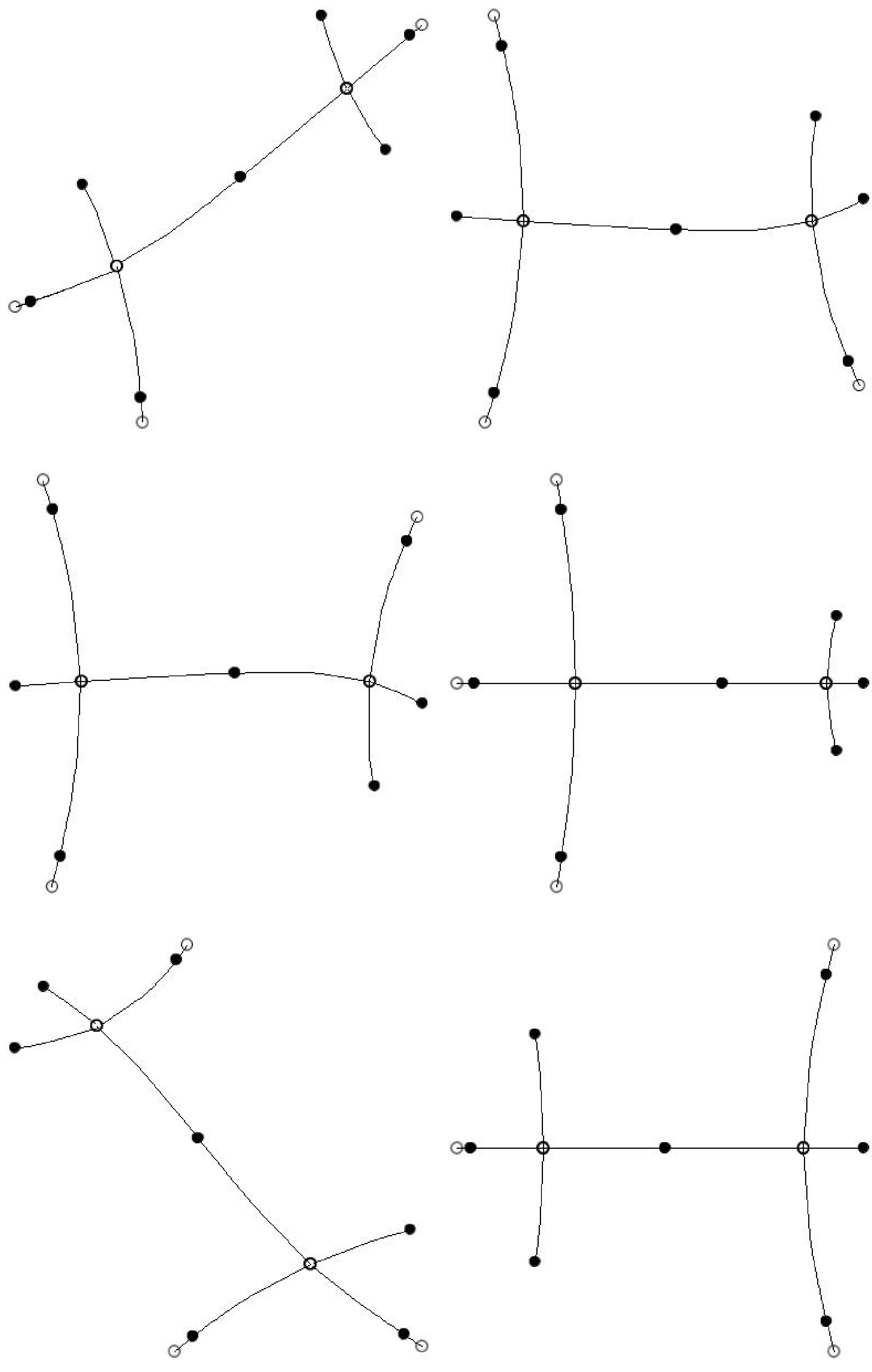
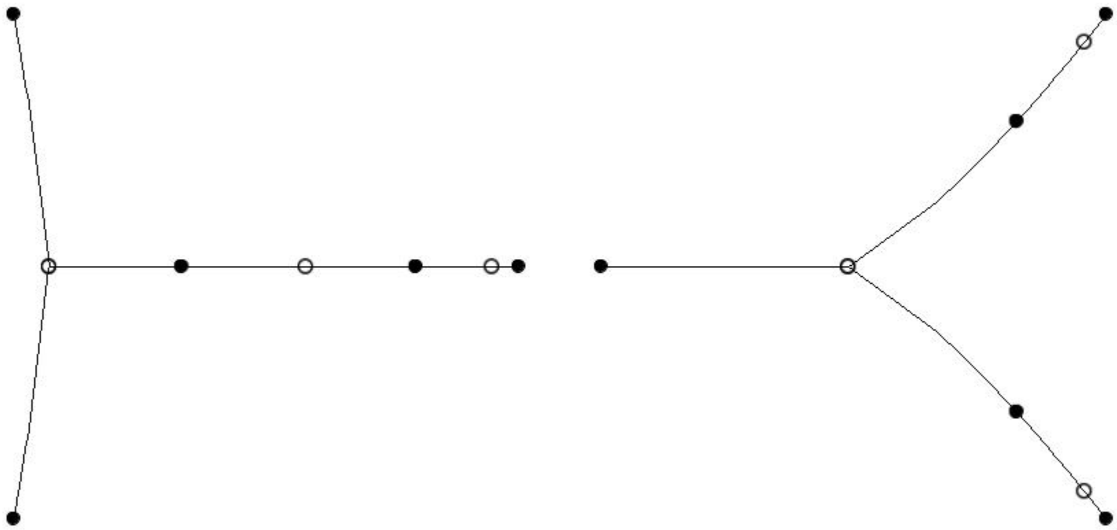


Figure 9. Another example: Trees with valencies 3, 2, 2 and 2, 2, 1, 1 ,1



(a) $P(z) = z^3(z^2 - 2z + \frac{34}{7} - \frac{6}{7}\sqrt{21})^2$

(b) $P(z) = z^3(z^2 - 2z + \frac{34}{7} + \frac{6}{7}\sqrt{21})^2$

References

[1] S. Lando, A. Zvonkin. *Graphs on Surfaces and Their Applications.* Encyclopaedia of Mathematical Sciences, vol. 141. Springer-Verlag, 2004

[2] J. Betrema, A. Zvonkin. *Plane Trees and Shabat Polynomials.* Discrete Mathematics, vol 153 (1996), pages 47-58

[3] L. Zapponi. *What is... a Dessin d'Enfant?* Notices of the AMS, vol 50, no. 7 (2003)

[4] H.E. Rose. *A Course on Finite Groups.* Springer-Verlag, 2009

[5] R. Guralnick, W. Kantor. *Probabilistic Generation of Finite Simple Groups.* Journal of Algebra, 234 (2000), pp. 743-792

[6] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, and R. Abbott. *ATLAS of Finite Group Representations.* `http://brauer.maths.qmul.ac.uk/Atlas/v3/`

[7] N. . Adrianov, Yu. Yu. Kochetkov, A. D. Suvorov, G. B. Shabat. *Mathieu Groups and Plane Trees.* Fundamentalnaya i prikladnaya matematika 1 (1995), 377-384.

School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005, Australia

E-mail address: `william.crawford@student.adelaide.edu.au`